Efficient Checking of Coherence and Propagation of Imprecise Probability Assessments

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Abstract

We consider the computational difficulties in the checking of coherence and propagation of imprecise probability assessments. We examine the linear structure of the random gain in betting criterion and we propose a general methodology which exploits suitable subsets of the set of values of the random gain. In this way the checking of coherence and propagation amount to examining linear systems with a reduced number of unknowns. We also illustrate an example.

Keywords: Coherent probability assessments, Propagation, Random gain, Computation, Algorithms.

1 Introduction

The probabilistic treatment of uncertainty by means of precise or imprecise probability assessments is well known. When the family of conditional events has no particular structure a suitable methodology is that based on the de Finetti's coherence principle, or generalizations of it. In this paper, we consider the computational difficulties connected with the checking of coherence. Then, exploiting the linear structure of the betting criterion, we propose a method to check coherence using suitable subsets of the set of values of the random gain. This amounts to study linear systems with a reduced number of unknowns. The paper is organized as follows. In section 2, after some comments related to the starting set of constituents, we recall some preliminary concepts and results. In section 3 we examine the linear structure of the betting criterion, with the aim of improving the efficiency of the procedure for checking coherence and for propagation. Finally, in section 4 we give an example.

2 Some remarks on constituents and betting criterion

Let $\mathcal{P}_n = (p_1, \ldots, p_n)$ be a precise probability assessment on a family $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$, where $J_n = \{1, \ldots, n\}$ and $p_i = P(E_i|H_i)$. We observe that $E_i|H_i = E_iH_i|H_i$ and then, to check coherence, we can start with the constituents generated by the family $\{E_iH_i, H_i, i \in J_n\}$. Then, as $E_iH_iH_i^c = \emptyset$, we can avoid to split H_i^c into $E_iH_i^c \vee E_i^cH_i^c$ and, by expanding the expression

$$\bigwedge_{i \in J_n} (E_iH_i \vee E_i^cH_i \vee H_i^c) \ , \qquad (1)$$

we obtain a set $\mathcal{C}$ of constituents with cardinality less than or equal to 3^n. This procedure has been proposed, e.g., in [3] and [4], where a geometrical approach to coherence has been adopted by associating with each constituent $C_h \subseteq H_0 = H_1 \vee \cdots \vee H_n$ a suitable point $Q_h = (q_{h1}, \ldots, q_{hn}) \in I\!R^n$. Then, denoting by the same symbols the events and their indicators, for each given $C_h \subseteq H_0$ the corresponding value g_h of the random gain $G = \sum_{i=1}^n s_iH_i(E_i - p_i)$, associated with the pair $(\mathcal{F}_n, \mathcal{P}_n)$, can be represented by the expression $g_h = G(Q_h) = \sum_{i=1}^n s_i(q_{hi} - p_i)$. We recall the following definition of generalized coherence (g-coherence) introduced in [1].

Definition 1 Given a vector $\mathcal{A}_n = (\alpha_1, \ldots, \alpha_n)$ of probability lower bounds $P(E_i|H_i) \geq \alpha_i$, $i \in J_n$, on a family $\mathcal{F}_n = \{E_1|H_1, \ldots, E_n|H_n\}$, the vector $\mathcal{A}_n$ is said g-coherent if and only if there exists a precise coherent assessment $\mathcal{P}_n = (p_1, \ldots, p_n)$ on $\mathcal{F}_n$, with $p_i = P(E_i|H_i)$ such that $p_i \geq \alpha_i$ for each $i \in J_n$.

The Definition 1 can also be applied to imprecise assessments like $\alpha_i \leq P(E_i|H_i) \leq \beta_i$, since each inequality $P(E_i|H_i) \leq \beta_i$ amounts to the inequality $P(E_i^c|H_i) \geq 1 - \beta_i$, where E_i^c denotes the contrary event of E_i.
We denote by $C_1, \ldots, C_m$ the constituents contained in H_0 and, for each constituent $C_r, r \in J_m$, we introduce a vector $V_r = (v_{r1}, \ldots, v_{rn})$, where for each $i \in J_n$ it is

$$v_{ri} = \begin{cases} 1, & \text{if } C_r \subseteq E_i H_i , \\ 0, & \text{if } C_r \subseteq E_i^c H_i , \\ \alpha_i , & \text{if } C_r \subseteq H_i^c . \end{cases} \tag{2}$$

We denote by $(\mathcal{S}_n)$ the following system, with nonnegative unknowns $\lambda_1, \ldots, \lambda_m$, associated with $(\mathcal{F}_n, \mathcal{A}_n)$

$$\begin{cases} \sum_{r=1}^m \lambda_r v_{ri} \geq \alpha_i, \;\; i \in J_n, \\ \sum_{r=1}^m \lambda_r = 1, \;\; \lambda_r \geq 0, \; r \in J_m. \end{cases} \tag{3}$$

We denote respectively by Λ and S the vector of unknowns and the set of solutions of the system (3). Moreover, for every j we denote by Γ_j the set of subscripts r such that $C_r \subseteq H_j$, by F_j the set of subscripts r such that $C_r \subseteq E_j H_j$ and by $\Phi_j(\Lambda)$ the linear function $\sum_{r \in \Gamma_j} \lambda_r$. We denote by I_0 the (strict) subset of J_n defined as

$$I_0 = \{j \in J_n : M_j = Max_{\Lambda \in S} \Phi_j(\Lambda) = 0\} \tag{4}$$

and by $(\mathcal{F}_0, \mathcal{A}_0)$ the pair associated with the set I_0. Then, based on the previous concepts, a suitable procedure ([4]) can be used to check the g-coherence of $\mathcal{A}_n$. The g-coherent extension of $\mathcal{A}_n$ to a further conditional event $E_{n+1}|H_{n+1}$ has been studied in [1] where, defining a suitable interval $[p_\circ, p^\circ] \subseteq [0,1]$, the following result has been obtained.

Theorem 1 Given a g-coherent imprecise assessment $\mathcal{A}_n = ([\alpha_i, \beta_i], i \in J_n)$ on the family $\mathcal{F}_n = \{E_i|H_i, \;\; i \in J_n\}$, the extension $[\alpha_{n+1}, \beta_{n+1}]$ of $\mathcal{A}_n$ to a further conditional event $E_{n+1}|H_{n+1}$ is g-coherent if and only if the following condition is satisfied

$$[\alpha_{n+1}, \beta_{n+1}] \cap [p_\circ, p^\circ] \neq \emptyset \, .$$

In [1] the computation of the values $p_\circ, p^\circ$ is made by an algorithm implemented with Maple V.

3 Checking of g-coherence and propagation

Given $(\mathcal{F}_n, \mathcal{A}_n)$ and a subset J of J_n, we denote by $(\mathcal{F}_J, \mathcal{A}_J)$ the pair associated with J and by $(\mathcal{S}_J)$ the corresponding system. Moreover, we denote by H_J the event $(\bigvee_{j \in J} H_j)$ and by G_J the following random gain, associated with $(\mathcal{F}_J, \mathcal{A}_J)$,

$$G_J \; = \; \sum_{j \in J} s_j H_j (E_j - \alpha_j) \, ,$$

where $s_j \geq 0$, for every $j \in J$. We observe that, in the case of an interval-valued assessment $\{[\alpha_i, \beta_i], i \in J_n\}$ the random gain G_J can be represented by the following expression

$$G_J \; = \; \sum_{j \in J} H_j [s_j (E_j - \alpha_j) - \sigma_j (E_j - \beta_j)] \, , \tag{5}$$

with $s_j \geq 0, \sigma_j \geq 0$, for every $j \in J$. It can be proved that $\mathcal{A}_n$ is g-coherent if and only if, for every $J \subseteq J_n$, the following condition is satisfied

$$Max \; G_J | H_J \geq 0 \, .$$

In particular, denoting by G_n the random gain associated with $(\mathcal{F}_n, \mathcal{A}_n)$, in order $\mathcal{A}_n$ be g-coherent the following (necessary) condition, equivalent to compatibility of the system (3), must be satisfied

$$Max \; G_n | H_0 \geq 0 \, . \tag{6}$$

Denoting by $\mathcal{G} = \{g_1, \ldots, g_m\}$ the set of possible values of $G_n | H_0$, for every subscript h the value g_h associated with the constituent C_h can be represented by the following expression

$$g_h \; = \; \sum_{i=1}^n s_i (v_{hi} - \alpha_i) \; = \; \sum_{i : C_h \subseteq H_i} s_i (v_{hi} - \alpha_i) \, . \tag{7}$$

Moreover, given three disjoint subsets J', J'', J''' of J_n, with $J' \cup J'' \cup J''' = J_n$, assume that there exist three constituents C_h, C_k, C_r such that

$$\begin{aligned} v_{hi} = \alpha_i \, , \quad & \text{for every} \quad i \in J' \cup J''', \\ v_{ki} = \alpha_i \, , \quad & \text{for every} \quad i \in J'' \cup J''', \\ v_{ri} = \alpha_i \, , \quad & \text{for every} \quad i \in J''', \end{aligned}$$

with

$$v_{hi} = v_{ri} \ , \quad \text{for every} \quad i \in J'',$$
$$v_{ki} = v_{ri} \ , \quad \text{for every} \quad i \in J'.$$

Then, for the corresponding values g_h, g_k, g_r we obtain

$$\begin{aligned} g_h &= \textstyle\sum_{i \in J''} s_i(v_{hi} - \alpha_i) \ = \ \sum_{i \in J''} s_i(v_{ri} - \alpha_i) \ , \\ g_k &= \textstyle\sum_{i \in J'} s_i(v_{ki} - \alpha_i) \ = \ \sum_{i \in J'} s_i(v_{ri} - \alpha_i) \ , \\ g_r &= \textstyle\sum_{i \in J' \cup J''} s_i(v_{ri} - \alpha_i) \ = \ g_h + g_k \ . \end{aligned}$$

Based on the above relation, we observe that the value of g_r is *not relevant* for the checking of condition (6) as

$$g_h < 0 \ , \ \ g_k < 0 \ \implies \ g_r < 0 \ ,$$

or conversely

$$g_r \geq 0 \ \implies \ g_h \geq 0 \ \text{ or } \ g_k \geq 0 \ .$$

By the same reasoning, if $g_r = a g_h + b g_k$, with $a > 0, b > 0$, then g_r is not relevant. As an example, denoting by V_h, V_k, V_r the vectors associated with C_h, C_k, C_r, if

$$V_r = x V_h + (1 - x) V_k, \ \ 0 < x < 1,$$

then

$$\begin{aligned} & g_r = G_n(V_r) = G_n[x V_h + (1 - x) V_k] = \\ & = x G_n(V_h) + (1 - x) G_n(V_k) = x g_h + (1 - x) g_k, \end{aligned}$$

so that g_r is not relevant. By the previous remarks, we have

Theorem 2 Given a subscript $r \in J_m$, if there exists a strict subset $\mathcal{T}_r$ of the set J_m, with $r \notin \mathcal{T}_r$, such that

$$g_r = \sum_{j \in \mathcal{T}_r} a_j g_j \ ; \ a_j > 0 \ , \ \forall j \in \mathcal{T}_r \ , \tag{8}$$

then g_r is not relevant.

More in general, we have

Theorem 3 Given a subscript $r \in J_m$, if there exist a strict subset $\mathcal{T}_r$ of the set J_m, with $r \notin \mathcal{T}_r$, and a positive constant c_r such that

$$g_r \ \leq \ c_r \mathop{Max}_{j \in \mathcal{T}_r} \ g_j \ , \tag{9}$$

then g_r is not relevant.

We remark that in general the constant c_r depends on the values $s_j, j \in J_n$. Then, by the previous results we obtain

Theorem 4 Given a strict subset $\mathcal{T}$ of the set J_m, if for every $r \notin \mathcal{T}$ there exist $T_r \subseteq \mathcal{T}$ and a positive constant c_r satisfying the condition (9), then the condition (6) is equivalent to the following one

$$\mathop{Max}_{j \in \mathcal{T}} \ g_j \ \geq \ 0 \ . \tag{10}$$

Based on suitable alternative theorems, the condition (10) is equivalent to the existence of a solution $(\lambda_1, \ldots, \lambda_m)$ of the system (3), such that $\lambda_r = 0$ for every subscript $r \notin \mathcal{T}$. Then, in order to check condition (10) we only need to study the compatibility of a system $\mathcal{S}_n^{\mathcal{T}}$, like (3), with a number of unknowns equal to the cardinality k of $\mathcal{T}$. In many cases k is drastically less than m. Therefore, to diminish the number of unknowns in the system (3), we need to examine the set $\mathcal{G}$ in order to determine a (possibly minimal) subset $\mathcal{T}$ of J_m satisfying, for all $r \notin \mathcal{T}$, the condition (9), with $\mathcal{T}_r \subseteq \mathcal{T}$. The checking of the g-coherence can be made by the following modified version of an algorithm proposed in [4].

Algorithm 1 Let be given $(J_n, \mathcal{F}_n, \mathcal{A}_n)$.

1. Determine a subset $\mathcal{T}$ which satisfies the condition (9), with $\mathcal{T}_r \subseteq \mathcal{T}$, for all $r \notin \mathcal{T}$;

2. Construct the system $(\mathcal{S}_n^{\mathcal{T}})$ and check its compatibility;

3. If the system $(\mathcal{S}_n^{\mathcal{T}})$ is not compatible then $\mathcal{A}_n$ is not g-coherent and the procedure stops; otherwise, replacing in Definition (4) the set of solutions of the system (3) by the set of solutions of the system $(\mathcal{S}_n^{\mathcal{T}})$, compute the set I_0;

4. If $I_0 = \emptyset$ then $\mathcal{A}_n$ is g-coherent and the procedure stops, otherwise set $(J_n, \mathcal{F}_n, \mathcal{A}_n) = (I_0, \mathcal{F}_0, \mathcal{A}_0)$ and repeat steps 1-3.

The extension of $\mathcal{A}_n$ can be made by a modified version of an algorithm proposed in [1]. Such algorithm, due to the lack of space, here is not included.

4 An example

Given the assessment $\mathcal{A}_3 = ([\frac{1}{5}, \frac{1}{4}], [\frac{1}{10}, \frac{1}{5}], [\frac{1}{10}, \frac{1}{4}])$ on $\mathcal{F}_3 = \{B|AC,\, C|(A \vee B), D|(B \vee C)\}$, we examine the g-coherence of $\mathcal{A}_3$ and its extension to $A|BCD^c$. By (1), we obtain the following 11 constituents contained in $H_0 = A \vee B \vee C$.

$$C_1 = ABCD,\; C_2 = ABCD^c,\; C_3 = BC^cD,$$
$$C_4 = BC^cD^c,\; C_5 = AB^cCD,\; C_6 = AB^cCD^c,$$
$$C_7 = AB^cC^c,\; C_8 = A^cBCD,\; C_9 = A^cBCD^c,$$
$$C_{10} = A^cB^cCD,\; C_{11} = A^cB^cCD^c.$$

We first observe that, based on (5), with $J = J_n = \{1, 2, 3\}$, one has

$$g_{10} < 0 \Longrightarrow g_{11} \geq 0\;, \quad g_{11} < 0 \Longrightarrow g_{10} \geq 0,$$

so that $Max\{g_{10}, g_{11}\} \geq 0$ and (6) is surely satisfied. Applying our method, we obtain

$$g_3 = g_7 + g_{10}\;, \quad g_4 = g_7 + g_{11}\;,$$

so that g_3 and g_4 are not relevant. Moreover, it is

$$g_1 = \delta_1 + g_8\;, \quad g_5 = \delta_2 + g_8\;,$$
$$g_2 = \delta_1 + g_9\;, \quad g_6 = \delta_2 + g_9\;,$$

with

$$\delta_1 = \frac{4}{5}s_1 - \frac{3}{4}\sigma_1\;, \quad \delta_2 = -\frac{1}{5}s_1 + \frac{1}{4}\sigma_1\;.$$

For every s_1, σ_1, the quantities δ_1, δ_2 cannot be both negative, so that

$$g_8 \leq Max\{g_1, g_5\}\;, \quad g_9 \leq Max\{g_2, g_6\}$$

and then g_8 and g_9 are not relevant. Therefore the subset $\mathcal{T} = \{1, 2, 5, 6, 7, 10, 11\}$ satisfies, for all $r \notin \mathcal{T}$, the condition (9), with $\mathcal{T}_r \subseteq \mathcal{T}$, so that the compatibility of (3), which has 11 unknowns, is equivalent to the compatibility of $\mathcal{S}_n^{\mathcal{T}}$ which has 7 unknowns. Then, by Algorithm 1, $\mathcal{A}_3$ is g-coherent. Given an assessment $\mathcal{A}_4$ on $\mathcal{F}_3 \cup \{A|BCD^c\}$, for the corresponding random gain the condition $g_9 \leq Max\{g_2, g_6\}$ is no more satisfied. Then, in the algorithm concerning the extension of $\mathcal{A}_3$ to $A|BCD^c$, the starting system has 8 (instead of 11) unknowns. It could be verified that $[p_\circ, p^\circ] = [0, 1]$.
Actually, deepening the analysis, one has

$$g_{10} \leq Max\{g_3, g_8\} \leq Max\{g_7{+}g_{10}, Max\{g_1, g_5\}\}$$

from which it follows that, if $g_1,\; g_5,\; g_7$ are negative, then g_{10} is negative too, so that it is not relevant. It could also be verified that

$$g_{11} \leq Max\{g_4, g_9\} \leq Max\{g_7{+}g_{11}, Max\{g_2, g_6\}\}$$

and then g_{11} is not relevant too. Therefore, we could apply the Algorithm 1 with $\mathcal{T} = \{1, 2, 5, 6, 7\}$. Differently from the "local" approach proposed in [2] to check coherence of *precise* probability assessments, our method for checking g-coherence and for propagation of *imprecise* conditional probability assessments is "global" and its efficient implementation is strictly connected with the choice of a good strategy for determining the subset $\mathcal{T}$. Work in progress concerns the application of our method to families of *conjunctive* conditional events, for which an efficient "global" procedure has been proposed in [5].

References

[1] Biazzo V., and Gilio A., A generalization of the fundamental theorem of de Finetti for imprecise conditional probability assessments, International Journal of Approximate Reasoning, 2000. Forthcoming.

[2] Capotorti A., Tognoloni S., and Vantaggi B., Local coherence, working paper, 1999.

[3] Gilio A., Conditional events and subjective probability in management of uncertainty, in *Uncertainty in Intelligent Systems* (Eds. B. Bouchon-Meunier, L. Valverde, R. R. Yager), Elsevier Science Publ. B. V., North-Holland, 1993, 109-120.

[4] Gilio A., Probabilistic consistency of conditional probability bounds, in *Advances in Intelligent Computing*, Lecture Notes in Computer Science 945 (B. Bouchon-Meunier, R. R. Yager, and L. A. Zadeh, Eds.), Springer-Verlag, Berlin Heidelberg, 200-209, 1995.

[5] Lukasiewicz T., Efficient Global Probabilistic Deduction from Taxonomic and Probabilistic Knowledge-Bases over Conjunctive Events, *Proc. of the 6th International Conference on Information and Knowledge Management*, ACM Press, 75-82, 1997.