Computational Aspects in Checking of Coherence and Propagation of Conditional Probability Bounds

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Abstract. In this paper we consider the problem of reducing the computational difficulties in g-coherence checking and propagation of imprecise conditional probability assessments. We review some theoretical results related with the linear structure of the random gain in the betting criterion. Then, we propose a modified version of two existing algorithms, used for g-coherence checking and propagation, which are based on linear systems with a reduced number of unknowns. The reduction in the number of unknowns is obtained by an iterative algorithm. Finally, to illustrate our procedure we give some applications.

1 Introduction

When facing real problems we often need to reason with uncertain information under partial knowledge. Then, the probabilistic treatment of uncertainty by means of precise or imprecise probability assessments is a well founded theoretical approach. Usually, the probabilistic assessments are defined on a given family of conditional events which has no particular algebraic structure. In these cases a suitable probabilistic methodology is that based on the coherence principle of de Finetti (see for example [1], [4], [7]), or on similar principles like that ones adopted for lower and upper probabilities ([10], [11]). As well known, the global checking of coherence is based on linear programming techniques which have an exponential complexity. An efficient global procedure for the probabilistic deduction from probabilistic knowledge-bases has been proposed in [8]. However, in the quoted paper it has been made the restrictive assumption that, for every probabilistic formula $(E|H)[\alpha, \beta]$ in the knowledge base, both E and H are conjunctive events and moreover $P(H) > 0$. Based on an idea suggested in [5], a promising procedure for a local checking of coherence of precise conditional probability assessments has been examined in a recent working paper ([3]). In this paper we consider the problem of reducing the computational difficulties in g-coherence checking and propagation of conditional probability bounds. We briefly review some theoretical results obtained in [2] which are related with the linear structure of the random gain in the betting criterion. Then, we propose a modified version of two algorithms given in [7] and [2], by means of which the g-coherence checking and propagation can be made by studying, in each step, the compatibility of a linear system with a reduced number of unknowns. The paper is organized as follows. In section 2 we recall some preliminary concepts and notations. In section 3 we first make some remarks concerning the set of constituents on which basis we can start the checking of g-coherence. Then, we briefly review some theoretical results ([2]), connected with the linear structure of the random gain in the betting criterion, which allow to reduce the number of unknowns in the linear systems used in our algorithms. In section 4 we give two algorithms for the checking of g-coherence and for the propagation of imprecise probability assessments. In section 5 we give an iterative algorithm to determine the reduced set of unknowns in our linear systems and we examine some aspects related to the computational complexity. In section 6 we give some applications of our iterative algorithm. Finally, in section 7 we give some conclusions and comments on specific aspects which need further work.

2 Preliminary concepts and notations

We recall some preliminary concepts which will be used in the next sections. Given a family $\mathcal{F}_n = \{E_1|H_1, \dots, E_n|H_n\}$ and a vector $\mathcal{A}_n = (\alpha_1, \dots, \alpha_n)$ of lower bounds $P(E_i|H_i) \geq \alpha_i$, with $i \in J_n = \{1, \dots, n\}$, we consider the following definition of generalized coherence (g-coherence) introduced in [1].

Definition 1 The vector of lower bounds $\mathcal{A}_n$ on $\mathcal{F}_n$ is said g-coherent if and only if there exists a precise coherent assessment $\mathcal{P}_n = (p_1, \dots, p_n)$ on $\mathcal{F}_n$, with $p_i = P(E_i|H_i)$, which is consistent with $\mathcal{A}_n$, that is such that $p_i \geq \alpha_i$ for each $i \in J_n$.

The Definition 1 can be also applied to imprecise assessments like

$$\alpha_i \leq P(E_i|H_i) \leq \beta_i \ , \ \ i \in J_n,$$

since each inequality $P(E_i|H_i) \leq \beta_i$ amounts to the inequality $P(E_i^c|H_i) \geq 1 - \beta_i$, where E_i^c denotes the contrary event of E_i.
Given the pair $(\mathcal{F}_n, \mathcal{A}_n)$, associated with the set J_n, denote by $\boldsymbol{P}$ the partition of the certain event Ω obtained by expanding the expression

$$\bigwedge_{i \in J_n} (E_i H_i \vee E_i^c H_i \vee H_i^c) \tag{1}$$

and by $C_1, \dots, C_m$ the atoms or constituents of $\boldsymbol{P}$ contained in $H_0 = \bigvee_{j \in J_n} H_j$. Moreover, we denote by C_0 the constituent H_0^c (if $H_0^c \neq \emptyset$), and in this case it is: $\boldsymbol{P} = \{C_0, C_1, \dots, C_m\}$. Of course, one has:

$$\bigvee_{C_h \in P} C_h = H_0 \vee H_0^c = \Omega \ .$$

In next section we will argue that the partition $\boldsymbol{P}$ is a convenient one to determine the starting system in the algorithm checking coherence. For each constituent $C_r, r = 1, \dots, m$, we introduce a vector $V_r = (v_{r1}, \dots, v_{rn})$, where for each $i \in J_n$ it is

$$v_{ri} \ = \begin{cases} 1 \ , & \text{if } C_r \subseteq E_i H_i \ , \\ 0 \ , & \text{if } C_r \subseteq E_i^c H_i \ , \\ \alpha_i \ , & \text{if } C_r \subseteq H_i^c \ . \end{cases} \tag{2}$$

Given an imprecise assessment $\mathcal{A}_n = (\alpha_1, \ldots, \alpha_n)$ on $\mathcal{F}_n$, we denote by $(\mathcal{S}_n)$ the following system with nonnegative unknowns $\lambda_1, \ldots, \lambda_m$.

$$\left\{ \begin{array}{l} \sum_{r=1}^{m} \lambda_r v_{ri} \geq \alpha_i, \;\; i \in J_n, \\ \sum_{r=1}^{m} \lambda_r = 1, \;\; \lambda_r \geq 0, \; r \in J_m. \end{array} \right. \tag{3}$$

We say that $(\mathcal{S}_n)$ is associated with the pair $(\mathcal{F}_n, \mathcal{A}_n)$.
In an analogous way, given a subset J of J_n, we denote by $(\mathcal{F}_J, \mathcal{A}_J)$ the pair corresponding to J and by $(\mathcal{S}_J)$ the system associated with $(\mathcal{F}_J, \mathcal{A}_J)$. Then, based on the previous concepts, a suitable procedure ([7]) can be used to check the g-coherence of $\mathcal{A}_n$. The g-coherent extension of $\mathcal{A}_n$ to a further conditional event $E_{n+1}|H_{n+1}$ has been studied in [1] where, defining a suitable interval $[p_\circ, p^\circ] \subseteq [0, 1]$, the following result has been obtained.

Theorem 1 Given a g-coherent imprecise assessment $\mathcal{A}_n = ([\alpha_i, \beta_i], i \in J_n)$ on the family $\mathcal{F}_n = \{E_i|H_i, \; i \in J_n\}$, the extension $[\alpha_{n+1}, \beta_{n+1}]$ of $\mathcal{A}_n$ to a further conditional event $E_{n+1}|H_{n+1}$ is g-coherent if and only if the following condition is satisfied

$$[\alpha_{n+1}, \beta_{n+1}] \cap [p_\circ, p^\circ] \neq \emptyset \; .$$

In the quoted paper an algorithm has been proposed to determine $[p_\circ, p^\circ]$.

3 Preliminary results on the checking of g-coherence and propagation

In this section we first give some preliminary remarks and then we briefly review some theoretical results obtained in [2]. A preliminary problem, related to the computational aspects, is that of suitably determining the initial set $\mathcal{C}$ of constituents, on which basis we can start the checking of coherence. Since it is irrelevant to consider precise or imprecise assessments, for the sake of simplicity, we examine the case of precise ones. Let be given a probability assessment $\mathcal{P}_n = (p_1, \ldots, p_n)$ on a family $\mathcal{F}_n = \{E_i|H_i, i = 1, \ldots, n\}$, with $p_i = P(E_i|H_i)$. We observe that $E_i|H_i = E_iH_i|H_i$ so that $\{E_i|H_i, i = 1, \ldots, n\} = \{E_iH_i|H_i, i = 1, \ldots, n\}$ and then, in the algorithms for the coherence checking and propagation of $\mathcal{P}_n$, we can start with the partition generated by the family $\{E_iH_i, H_i, i = 1, \ldots, n\}$, instead of the family $\{E_i, H_i, i = 1, \ldots, n\}$. Then, as $E_iH_iH_i^c = \emptyset$, the constituents of $\mathcal{C}$ are obtained by expanding the expression (1), so that its cardinality is less than or equal to 3^n. Given an imprecise assessment $\mathcal{A}_n$ on the family $\mathcal{F}_n$, for every $J \subseteq J_n$, we denote by H_J the event $(\bigvee_{j \in J} H_j)$ and by G_J the following random quantity (which in the betting scheme can be interpreted as a random gain)

$$G_J = \sum_{j \in J} s_j H_j (E_j - \alpha_j),$$

where $s_j \geq 0$, for every $j \in J$. We say that G_J is associated with the pair $(\mathcal{F}_J, \mathcal{A}_J)$. We observe that, in the case of an interval-valued assessment

$\{[\alpha_i, \beta_i], i \in_n\}$ the random gain G_J can be represented by the following expression

$$G_J = \sum_{j \in J} H_j[s_j(E_j - \alpha_j) - \sigma_j(E_j - \beta_j)], \tag{4}$$

with $s_j \geq 0, \sigma_j \geq 0$, for every $j \in J$. It can be proved that ([2]) $\mathcal{A}_n$ is g-coherent if and only if, for every $J \subseteq J_n$, the following condition is satisfied

$$Max\ G_J|H_J \geq 0.$$

In particular, denoting by G_n the random gain associated with the pair $(\mathcal{F}_n, \mathcal{A}_n)$, in order $\mathcal{A}_n$ be g-coherent the following condition must be satisfied

$$Max\ G_n|H_0 \geq 0. \tag{5}$$

Based on a suitable alternative theorem, the above condition is equivalent to the compatibility of the system (3). Denoting by $\mathcal{G} = \{g_1, \ldots, g_m\}$ the set of possible values of $G_n|H_0$, for every subscript h the value g_h associated with the constituent C_h can be represented by the following expression

$$g_h = \sum_{i=1}^{n} s_i(v_{hi} - \alpha_i) = \sum_{i: C_h \subseteq H_i} s_i(v_{hi} - \alpha_i). \tag{6}$$

Denoting by $\mathbf{z}$ the vector $(z_1, \ldots, z_n)$ and by $f(\mathbf{z})$ the linear function

$$f(z_1, \ldots, z_n) = \sum_{i=1}^{n} s_i z_i,$$

for every $h = 1, \ldots, m$, we have

$$g_h = G(V_h) = f(V_h - \mathcal{A}_n). \tag{7}$$

Remark 1 Notice that, given two distinct constituents C_h, C_k, it may be $V_h = V_k$. Then one has $g_h = g_k$, so that g_k is not relevant. Therefore, to reduce the number of unknowns, a preliminary computation of the set of distinct vectors V_h's is suitable.

We now illustrate the basic idea which allows to reduce the number of unknowns in the linear systems involved in the algorithms for checking g-coherence and for propagation of lower and upper probability bounds. Given a partition $\{J', J'', J'''\}$ of J_n, i.e. three disjoint subsets J', J'', J''', with $J' \cup J'' \cup J''' = J_n$, assume that there exist three constituents C_1, C_2, C_3 such that

$$\begin{array}{ll} C_1 \subseteq H_i^c\ , & \forall\, i \in J' \cup J'''\ , \\ C_2 \subseteq H_i^c\ , & \forall\, i \in J'' \cup J'''\ , \\ C_3 \subseteq H_i^c\ , & \forall\, i \in J'''\ . \end{array}$$

Based on (2), for the corresponding vectors V_1, V_2, V_3 one has

$$\begin{array}{ll} v_{1i} = \alpha_i, & \text{for every } \ i \in J' \cup J''', \\ v_{2i} = \alpha_i, & \text{for every } \ i \in J'' \cup J''', \\ v_{3i} = \alpha_i, & \text{for every } \ i \in J''', \end{array}$$

with

$$\begin{array}{ll} v_{1i} = v_{3i}, & \text{for every } \ i \in J'', \\ v_{2i} = v_{3i}, & \text{for every } \ i \in J'. \end{array}$$

Then, concerning the values g_1, g_2, g_3 associated with C_1, C_2, C_3, from (6) we obtain

$$\begin{array}{l} g_1 = \sum_{i \in J''} s_i(v_{1i} - \alpha_i) = \sum_{i \in J''} s_i(v_{3i} - \alpha_i), \\ g_2 = \sum_{i \in J'} s_i(v_{2i} - \alpha_i) = \sum_{i \in J'} s_i(v_{3i} - \alpha_i), \\ g_3 = \sum_{i \in J' \cup J''} s_i(v_{3i} - \alpha_i) = g_1 + g_2. \end{array}$$

Based on the above relation, we observe that the value of g_3 is *not relevant* for the checking of the condition (5) as

$$g_1 < 0, \ g_2 < 0 \implies g_3 < 0$$

and, conversely,

$$g_3 \geq 0 \implies g_1 \geq 0 \ \text{ or } \ g_2 \geq 0.$$

By the same reasoning, if there exist three constituents C_1, C_2, C_3 and two positive numbers a, b such that

$$g_3 \leq a g_1 + b g_2,$$

then g_3 is not relevant for checking coherence. By the previous remarks, defining $J_m = \{1, \ldots, m\}$, we have

Theorem 2 Given a subscript $r \in J_m$, if there exists a strict subset $\mathcal{T}_r$ of the set J_m, with $r \notin \mathcal{T}_r$, such that

$$g_r \leq \sum_{j \in \mathcal{T}_r} a_j g_j \ ; \ a_j > 0 \ , \ \forall j \in \mathcal{T}_r \ ,$$

then g_r is not relevant.

More in general, we have

Theorem 3 Given a subscript $r \in J_m$, if there exist a strict subset $\mathcal{T}_r$ of the set J_m, with $r \notin \mathcal{T}_r$, and a positive constant c_r such that

$$g_r \ \leq \ c_r Max \ \{g_j\}_{j \in \mathcal{T}_r} \tag{8}$$

then g_r is not relevant.

By the previous results we obtain

Theorem 4 Let $\mathcal{T}$ be a strict subset of the set J_m such that for every $r \notin \mathcal{T}$ there exist $T_r \subseteq \mathcal{T}$ and a positive constant c_r which satisfy the condition (8). Then

$$Max \ \{g_j\}_{j \in J_m} \geq 0 \iff Max \ \{g_j\}_{j \in \mathcal{T}} \geq 0 \ .$$

Based again on an alternative theorem ([2]), a result equivalent to the Theorem 4 is the following one.

Theorem 5 Let $\mathcal{T} \subset J_m$ be such that for every $r \notin \mathcal{T}$ there exist $T_r \subseteq \mathcal{T}$ and a positive constant c_r which satisfy the condition (8). Then the condition (5) is satisfied if and only if there exists a solution $(\lambda_1, \ldots, \lambda_m)$ of the system $(\mathcal{S}_n)$, such that $\lambda_r = 0$, $\forall r \notin \mathcal{T}$.

Then, denoting by k, with $k < m$, the cardinality of $\mathcal{T}$, in order to check condition (5) we only need to study the compatibility of a system like (3), denoted by $\mathcal{S}_n^{\mathcal{T}}$, which has k (instead of m) unknowns. Therefore, to reduce the number of unknowns in the system (3), we need to examine the set $\mathcal{G}$ in order to determine a (possibly minimal) subset $\mathcal{T}$ of J_m satisfying, for all $r \notin \mathcal{T}$, the condition (8), with $\mathcal{T}_r \subseteq \mathcal{T}$.

4 Modified algorithms for checking g-coherence and propagation

We denote respectively by Λ and S the vector of unknowns and the set of solutions of the system $(\mathcal{S}_n^{\mathcal{T}})$. Moreover, for every j we denote by Γ_j the set of subscripts r such that $C_r \subseteq H_j$ and by F_j the set of subscripts r such that $C_r \subseteq E_j H_j$. For each j, we define the linear function

$$\Phi_j(\Lambda) = \sum_{r \in \Gamma_j} \lambda_r.$$

Moreover, we denote by I_0 the (strict) subset of J_n defined as

$$I_0 = \{j \in J_n : M_j = Max_{\Lambda \in S} \Phi_j(\Lambda) = 0\} \tag{9}$$

and by $(\mathcal{F}_0, \mathcal{A}_0)$ the pair associated with the set I_0. Then, the checking of the g-coherence can be made by the following modified version of an algorithm proposed in [7]. An algorithm for the determination of the subset $\mathcal{T}$ is given in the next section.

Algorithm 1 Let be given the tern $(J_n, \mathcal{F}_n, \mathcal{A}_n)$.

1. Determine a subset $\mathcal{T}$ satisfying, for all $r \notin \mathcal{T}$, the condition (8), with $\mathcal{T}_r \subseteq \mathcal{T}$;

2. Construct the system $(\mathcal{S}_n^{\mathcal{T}})$ and check its compatibility;

3. If the system $(\mathcal{S}_n^{\mathcal{T}})$ is not compatible then $\mathcal{A}_n$ is not g-coherent and the procedure stops; otherwise, compute the set I_0;

4. If $I_0 = \emptyset$ then $\mathcal{A}_n$ is g-coherent and the procedure stops, otherwise set $(J_n, \mathcal{F}_n, \mathcal{A}_n) = (I_0, \mathcal{F}_0, \mathcal{A}_0)$ and repeat steps 1-3.

Concerning the g-coherent extensions of $\mathcal{A}_n$, the computation of the values p_0, p^0 can be carried out by means of the following modified version of the algorithm proposed in [1].

Algorithm 2 Let be given the pair $(\mathcal{F}_n, \mathcal{A}_n)$, the conditional event $E_{n+1}|H_{n+1}$, and define $J_{n+1} = \{1, \ldots, n+1\}$.

1. Expand the expression
$$\bigwedge_{j \in J_{n+1}} \left(E_j H_j \vee E_j^c H_j \vee H_j^c\right),$$
and construct the vectors $V_1, \ldots, V_m$. Then, determine a subset $\mathcal{T}$ satisfying, for all $r \notin \mathcal{T}$, the condition (8), with $\mathcal{T}_r \subseteq \mathcal{T}$;

2. Construct the following system $(\mathcal{S}_{n+1}^{\mathcal{T}})$ in the unknowns $p_{n+1}, \lambda_h, h \in \mathcal{T}$:
$$\left\{ \begin{array}{l} \sum_{r \in F_{n+1} \cap \mathcal{T}} \lambda_r = p_{n+1} \sum_{r \in \Gamma_{n+1} \cap \mathcal{T}} \lambda_r \ , \\ \alpha_j \leq \sum_{r \in \mathcal{T}} v_{rj} \lambda_r \leq \beta_j, \ j \in J_n, \\ \sum_{r \in \mathcal{T}} \lambda_r = 1, \\ \lambda_r \geq 0, \ r \in \mathcal{T}. \end{array} \right. \tag{10}$$

3. Check the compatibility of system (10) under the condition $p_{n+1} = 0$ (respectively $p_{n+1} = 1$). If the system (10) is not compatible go to Step 3, otherwise go to Step 4;

4. Solve the following linear programming problem
$$\text{Compute :} \qquad \gamma' = Min \sum_{r \in F_{n+1} \cap \mathcal{T}} \lambda_r$$
$$\text{(respectively :} \qquad \gamma'' = Max \sum_{r \in F_{n+1} \cap \mathcal{T}} \lambda_r \ )$$
subject to:
$$\begin{array}{l} \alpha_j \leq \sum_{r \in \mathcal{T}} v_{rj} \lambda_r \leq \beta_j, \ j \in J_n, \\ \sum_{r \in \Gamma_{n+1} \cap \mathcal{T}} \lambda_r = 1, \ \lambda_r \geq 0, \ r \in \mathcal{T}. \end{array}$$
The minimum γ' (respectively the maximum γ'') of the *objective function* coincides with p_0 (respectively with p^0) and the procedure stops;

5. For each subscript j, compute the maximum M_j of the function Φ_j, subject to the constraints given by the system (10) with $p_{n+1} = 0$ (respectively $p_{n+1} = 1$). We have the following three cases:
   (a) $M_{n+1} > 0$;
   (b) $M_{n+1} = 0$, $M_j > 0$ for every $j \neq n+1$;
   (c) $M_j = 0$ for $j \in I_0 = J \cup \{n+1\}$, with $J \neq \emptyset$.

   In the first two cases it is $p_0 = 0$ (respectively $p^0 = 1$) and the procedure stops.
   In the third case, defining $I_0 = J \cup \{n+1\}$, set $J_{n+1} = I_0$ and $(\mathcal{F}_n, \mathcal{P}_n) = (\mathcal{F}_J, \mathcal{P}_J)$; then go to Step 1.

The procedure ends in a finite number of cycles by computing the value p_0 (respectively p^0).

5 Computation of the subset $\mathcal{T}$

In this section we will concentrate on the determination of a subset $\mathcal{T}$. We first observe that, in order to determine $\mathcal{T}$, we can use an iterative procedure, as shown by the following result.

Theorem 6 Assume that there exist two subsets $\mathcal{T}_1, \mathcal{T}_2$, with $\mathcal{T}_2 \subset \mathcal{T}_1 \subset J_m$, such that:

1. for every $r \in J_m \setminus \mathcal{T}_1$ there exist $T_r \subseteq \mathcal{T}_1$ and a positive constant c_r which satisfy (8) ;

2. for every $r \in \mathcal{T}_1 \setminus \mathcal{T}_2$, there exist $T_r \subseteq \mathcal{T}_2$ and a positive constant c_r which satisfy (8) .

Then, for every $r \in J_m \setminus \mathcal{T}_2$ there exist $T_r \subseteq \mathcal{T}_2$ and a positive constant c_r which satisfy (8) and therefore

$$Max\ G_n|H_0 \geq 0 \iff Max\{g_j\}_{j\in\mathcal{T}_2} \ \geq \ 0\ .$$

Remark 2 We observe that, based on the result above, we can determine a subset $\mathcal{T}_1$ such that, for every $r \in J_m \setminus \mathcal{T}_1$, there exist $h \in \mathcal{T}_1, k \in \mathcal{T}_1$, such that $g_r \leq g_h + g_k$. By iterating this procedure we can determine $\mathcal{T}_2 \subset \mathcal{T}_1$ such that, for every $k \in \mathcal{T}_1 \setminus \mathcal{T}_2$, there exist $s \in \mathcal{T}_2, t \in \mathcal{T}_2$, such that $g_k \leq g_s + g_t$. In this way, we automatically detect the cases of non relevance like $g_r \leq g_h + g_s + g_t$, when there exists a gain g_k such that $g_k \leq g_h+g_s$, or $g_k \leq g_h+g_t$, or $g_k \leq g_s+g_t$. If such a gain g_k doesn't exist, to establish the non relevance of g_r we need to explicitly check if the inequality $g_r \leq g_h+g_s+g_t$ holds for some tern (g_h, g_s, g_t); and so on.

In what follows we deepen in particular the aspect of detecting the cases like $g_r \leq g_h + g_k$ and we give an iterative algorithm to work out this problem. We also examine some aspects related to the computational complexity.
Based on (7), we observe that

$$g_r = f(V_r - \mathcal{A}_n), \quad g_h = f(V_h - \mathcal{A}_n), \quad g_k = f(V_k - \mathcal{A}_n),$$

and then, recalling that the coefficients $s_i, i \in J_n$, in the function f are nonnegative, one has

$$V_r - \mathcal{A}_n \ \leq \ V_h - \mathcal{A}_n + V_k - \mathcal{A}_n \implies g_r \ \leq \ g_h + g_k\ ,$$

that is

$$\mathcal{A}_n \ \leq \ V_h + V_k - V_r \implies g_r \ \leq \ g_h + g_k\ . \tag{11}$$

Based on (11), a procedure to determine a sequence of subsets $\mathcal{T}_1, \ldots, \mathcal{T}_i$, with $\mathcal{T}_1 \supset \mathcal{T}_2 \supset \cdots \supset \mathcal{T}_{i-1} = \mathcal{T}_i$, is given in the Algorithm 3. We observe that the sequence $\mathcal{T}_1, \ldots, \mathcal{T}_i$ may change if we reorder the sequence of vectors $V_1, \ldots, V_m$.

Algorithm 3 Let be given the tern $(J_n, \mathcal{F}_n, \mathcal{A}_n)$.

1. Expand the expression
$$\bigwedge_{j \in J_n} \left(E_j H_j \vee E_j^c H_j \vee H_j^c \right) \ .$$
Denote by $C_1, \ldots, C_s$ the constituents contained in H_0 and by $V_1, \ldots, V_s$ the corresponding vectors.

2. Denote by $\{V_1, \ldots, V_m\}$, where $m \leq s$, the set of distinct vectors and set $\mathcal{T}_0 = J_m$.

3. Set $\mathcal{T}_1 = \mathcal{T}_1^c = \emptyset$ and $r = 1$.

4. If $r \in \mathcal{T}_1$, then go to Step 6. Otherwise, go to Step 5.

5. If a subset $\{h, k\} \subseteq \mathcal{T}_0 \setminus \{\mathcal{T}_1^c \cup \{r\}\}$ is found such that (11) holds, then replace $\mathcal{T}_1^c$ by $\mathcal{T}_1^c \cup \{r\}$ and $\mathcal{T}_1$ by $\{\mathcal{T}_1 \cup \{h, k\}\}$. Otherwise, replace the set $\mathcal{T}_1$ by the set $\mathcal{T}_1 \cup \{r\}$.

6. If $r = |\mathcal{T}_0|$, go to Step 7. Otherwise, replace r by $r + 1$ and go to Step 4.

7. If $\mathcal{T}_1 \subset \mathcal{T}_0$, then introduce two sets $\mathcal{T}_2, \mathcal{T}_2^c$ and set $(\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_1^c) = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_2^c)$. Then, go to Step 3. Otherwise, $\mathcal{T} = \mathcal{T}_1$ and procedure ends.

The algorithm stops when, for some i, one has $\mathcal{T}_{i-1} = \mathcal{T}_i$ and $\mathcal{T} = \mathcal{T}_i$.
Of course, the previous algorithm is not the optimal one to determine the minimal subset $\mathcal{T}$ such that, for every $r \in J_m \setminus \mathcal{T}$ there exist $T_r \subseteq \mathcal{T}$ and a positive constant c_r which satisfy (8). For example, we could try to reduce the cardinality of set $\mathcal{T}$ obtained by the Algorithm 3, by checking for every subset $\{h, k, s, r\} \subseteq \mathcal{T}$ if the condition $g_r \leq g_h + g_k + g_s$ is satisfied, in which case g_r would be not relevant; and so on. Another modification of the algorithm could be that of trying to eliminate some g_r's during the process of generation of the constituents and not after that such generation is completed. The improvement of the Algorithm 3 requires further work.
Concerning the computational complexity of the Algorithm 3, we observe that for each $r \in J_m$ we should search for a subset $\{h, k\}$ satisfying the condition (11). Then, in the (worst) case in which all the gains g_r are relevant, as $r \in J_m = \{1, 2, \ldots, m\}$ and the subsets $\{h, k\}$ in $J_m \setminus \{r\}$ are $\frac{(m-1)(m-2)}{2}$, the number of times the algorithm will execute the Steps 3-7 is $\frac{m(m-1)(m-2)}{2}$. We must also take into account that the vectors V_h's have n components. Therefore, concerning this aspect, the complexity is $\mathcal{O}(nm^3)$.

6 Some applications

In this section we examine some applications of the Algorithm 3.

Example 1 We examine a simple example given in [8], where an efficient global procedure has been proposed to propagate conditional probability bounds for families of *conjunctive* conditional events (as in the family $\mathcal{F}_3$ below). In [9] the procedure has been generalized to the case of *conditioning events* (possibly) having probability zero. Given the vector of upper bounds $\mathcal{B}_3 = (0.2, 0.2, 0.2)$ on the family $\mathcal{F}_3 = \{B|A, C|AB, D|C\}$, let us consider the extension of $\mathcal{B}_3$ to $BCD|A$. The constituents contained in $H_0 = A \vee C$ are respectively

$$\begin{array}{llll} C_1 = ABCD & C_2 = ABCD^c & C_3 = ABC^c & C_4 = AB^cC^c \\ C_5 = A^cCD & C_6 = A^cCD^c & C_7 = AB^cCD & C_8 = AB^cCD^c \end{array}$$

As a preliminary remark, we could verify that g_4 is surely nonnegative and so the condition (5) is surely satisfied. Applying the Algorithm 3, we have $\mathcal{T}_0 = J_8$ and $\mathcal{T}_1 = \{2, 3, 4, 6\}$. In fact

$$g_1 \leq g_2 + g_4, \;\; g_r \leq g_4 + g_6, \;\; r \in \{5, 7, 8\}.$$

By iterating, one has $\mathcal{T}_2 = \mathcal{T}_3 = \mathcal{T} = \{3, 4, 6\}$. In fact $g_2 \leq g_3 + g_6$. Then, applying the Algorithm 1 we obtain that $\mathcal{A}_3$ is g-coherent.
Concerning the propagation of $\mathcal{B}_3$ to $BCD|A$, we observe that the constituents are the same. Moreover, given the assessment

$$P(B|A) \leq 0.2, \;\; P(C|AB) \leq 0.2, P(D|C) \leq 0.2, \;\; P(BCD|A) = p,$$

based on the Theorem 1 we must determine the interval $[p_\circ, p^\circ]$ of the values p which are coherent extensions of $\mathcal{B}_3$ to $BCD|A$. By slightly modifying the Algorithm 3 we have $\mathcal{T}_0 = J_8$ and $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T} = \{1, 3, 5, 6, 8\}$. In fact

$$g_2 \leq g_3 + g_6, \;\; g_4 = g_6 + g_8, \;\; g_7 \leq g_5 + g_8.$$

Notice that in [8] a set $\mathcal{T}$ with 6 (instead of 5) elements is obtained. Thus, the starting system of the Algorithm 2 has 5 (instead of 8) unknowns. We obtain $[p_0, p^0] = [0, 0.04]$.

Example 2 Given the imprecise assessment $\mathcal{A}_3 = (\frac{1}{5}, \frac{1}{10}, \frac{1}{10})$ on the family $\mathcal{F}_3 = \{B|AC, \, C|(A \vee B), D|(B \vee C)\}$, let us consider the extension of $\mathcal{A}_3$ to the conditional event $A|BCD^c$. We first examine the g-coherence of $\mathcal{A}_3$. In our case, by expanding the expression (1), we obtain the following 11 constituents contained in $H_0 = A \vee B \vee C$.

$$\begin{array}{llll} C_1 = ABCD & C_2 = ABCD^c & C_3 = BC^cD & C_4 = BC^cD^c \\ C_5 = AB^cCD & C_6 = AB^cCD^c & C_7 = AB^cC^c & C_8 = A^cBCD \\ C_9 = A^cBCD^c & C_{10} = A^cB^cCD & C_{11} = A^cB^cCD^c & \end{array}$$

We observe at first that, based on the formula (4), with $J = J_n = \{1, 2, 3\}$, it could be verified that

$$g_{10} < 0 \Longrightarrow g_{11} \geq 0, \;\; g_{11} < 0 \Longrightarrow g_{10} \geq 0,$$

so that $Max\{g_{10}, g_{11}\} \geq 0$ and therefore the condition (5) is surely satisfied. Applying the Algorithm 3, we have $\mathcal{T}_0 = J_{11}$ and $\mathcal{T}_1 = \{1, 5, 8\}$. In fact

$$g_2 \leq g_1 + g_8, \;\; g_r \leq g_1 + g_5, \;\; r \in \{3, 4, 6, 7, 9, 10, 11\}.$$

Iterating the algorithm, as $g_5 \leq g_1 + g_8$, we obtain $\mathcal{T}_2 = \{1, 8\}$. By iterating again, one has $\mathcal{T}_3 = \mathcal{T}_2$ and then $\mathcal{T} = \{1, 8\}$. Then, applying the Algorithm 1 we obtain that $\mathcal{A}_3$ is g-coherent.
Concerning the extension of $\mathcal{A}_3$ to $A|BCD^c$, by the (modified) Algorithm 3 we have $\mathcal{T}_0 = J_{11}$ and $\mathcal{T}_1 = \{1, 2, 5, 9\}$. In fact

$$g_r \leq g_1 + g_5, \;\; r \in \{3, 4, 6, 7, 8, 10, 11\}.$$

At the second iteration, one has $\mathcal{T}_2 = \mathcal{T}_1$ and then $\mathcal{T} = \{1, 2, 5, 9\}$. Thus, the starting system of the Algorithm 2 has 4 (instead of 11) unknowns. We obtain $[p_0, p^0] = [0, 1]$.

Example 3 Let $\mathcal{F}_{10}$ be the family

$$\{B_1|B_2, B_2|B_3, B_3|B_4, B_4|B_5, B_5|B_6, B_6^c|B_5, B_5^c|B_4, B_4^c|B_3, B_3^c|B_2, B_2^c|B_1\}$$

and $\mathcal{A}_{10}$ be the vector of lower bounds

$$(0.9, 0.9, 0.9, 0.9, 0.9, 0.8, 0.8, 0.8, 0.8, 0.8)$$

on $\mathcal{F}_{10}$. The constituents obtained by expanding 1 are 63. Applying the Algorithm 3 it is possible to reduce the set J_{63} to a subset $\mathcal{T}$ with 21 elements. If we consider the extension of $\mathcal{A}_{10}$ to $B_6|B_1$ applying the (modified) Algorithm 3, we obtain a subset $\mathcal{T}$ with 31 elements and it can be verified that $[p_0, p^0] = [0, 0.000542]$ (as in [8]).

7 Conclusions

In this paper, with the aim of reducing the computational difficulties, we have proposed a modified version of two existing algorithms for g-coherence checking and propagation of conditional probability bounds. Our algorithms, in each step, check the compatibility of a linear system with a reduced number of unknowns. At each step, to determine the set $\mathcal{T}$, we have proposed an iterative algorithm. As shown by the examples, the Algorithm 3, even if not optimal, produces a good reduction in the number of unknowns (perhaps, in some cases such reduction may be drastic). A modification which seems suitable is that of eliminating the unknowns during the process of generation of the constituents. Also an integration of local and global approaches could be useful. Further improvements of the Algorithm 3 are under study.

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