6^{th} Workshop on Uncertainty Processing WUPES'2003

Hejnice, Czech Republic
24 - 27th September, 2003

[Topics] [Programme committee] [Important Dates] [Venue]] [Location]
[Call for papers] [Conference fee] [Student grants] [Paper preparation]
[Information request] [On-line proceedings] [Registration & transport]

A series of Workshops on Uncertainty Processing (WUPES) was held in the Czech Republic in 1988, 1991, 1994, 1997, and 2000. Like the previous meetings the forthcoming Workshop will foster creative intellectual activities and the exchange of ideas in an informal atmosphere. Therefore we will keep the number of participants limited (about 40).

Topics

Contributions belonging to the various fields of uncertainty processing are invited. Typical examples are:

- probabilistic modelling (conditional independence models, graphical models, Bayesian networks, models based on coherence principles),
- logical and algebraic modelling (including fuzzy approaches),
- possibilistic approaches,
- models based on belief functions,
- representative applications.

The working character of the meeting is stressed by the fact that we also welcome papers casting new problems and inspiring discussion as well as contributions presenting promising but as yet not finished results.

Programme committee

Didier Dubois	(France)
Petr Hajek	(Czech Republic)
Radim Jirousek	(Czech Republic)
Gernot D. Kleiter	(Austria)
Peter Naeve	(Germany)
Romano Scozzafava	(Italy)
Milan Studeny	(Czech Republic)

Important Dates

15th May 2003	submission of extended abstracts (two pages) by e-mail to vejnar@vse.cz
15th June 2003	author notification of accepted papers
31st July 2003	camera-ready copy of accepted papers

Venue

The conference will be held in Hejnice - a small town situated in a beautiful natural setting. Ridges of the Jizera Mountains border the baroque cathedral of Visitation of our Lady. The workshop will take place in the ancient Franciscan cloister, where the Litomerice bishopric has created the International Center for Spiritual Rehabilitation. For more information on Hejnice visit the home page of the International Center for Spiritual Rehabilitation in Hejnice .

Conference fee

Conference fee (including registration, accommodation and full boarding) is 280 EUR (7840 CZK). It should be payed (preferably) by a bank transfer to:

Czech Society for Cybernetics and Informatics
Pod vodarenskou vezi 2
182 00 Praha 8
Czech Republic

Account number: 0208673319 / 0800
Reference: you must indicate your first name, surname and WUPES'03 as payment reference.
Czech participants should add "konstantni symbol: 0308" and "variabilni symbol: 230903"

Another possibility is to pay cash on site.

Student grants
Thanks to the support of EUNITE, five students are offered to attend the workshop "free of charge". Do not hesitate to contact us at vejnar@vse.cz.

Paper preparation
The PDF version of full papers (recommeded length is 8 - 12 pages) should be submitted by e-mail to vejnar@vse.cz before August 1. When preparing your contribution, please, follow the instructions for the authors. We recommend you to use our LaTex template. See files 6wupes.pdf and 6wupes.tex.

Information request
If you would like to be informed about the workshop, please, send an e-mail to vejnar@vse.cz.

TABLE OF CONTENTS

Logical conditions for coherent qualitative and numerical probability assessments 1
Veronica Biazzo, Angelo Gilio, Giuseppe Sanfilippo

Conditional exchangeability and locally strong coherence 13
Andrea Capotorti, Sara Gardi

On the optimal test for cross-classified multinomial distributions 25
Robert Castelo

Conditional probability as a function of the conditioning event: characterization of coherent enlargements 35
Simona Ceccacci, Chiara Morici, Tania Paneni

Inference in hybrid Bayesian networks with mixtures of truncated exponentials 47
Barry R. Cobb, Prakash P. Shenoy

Toward a general theory of conditional beliefs 65
Giulianella Coletti, Romano Scozzafava

Transformations of belief functions to probabilities 77
Milan Daniel

Belief functions applied to object recognition and object classification 91
Arthur P. Dempster, Wai Fung Chiu

Independence and conditional possibilities for strict t-norms 101
Laura Ferracutti, Barbara Vantaggi

Two generalizations of rough sets and their fundamental properties . 113
Masahiro Inuiguchi

Construction of implicational quantifiers from fuzzy implications 125
Jiří Ivánek

A numerical method for learning Bayesian networks from statistical data 133
Martin Janžura, Jan Nielsen

Enumerating essential graphs 135
Gernot D. Kleiter

On learning graphical models with hidden variables 149
Tomáš Kočka

Possibilistic laws of large numbers 159
Ivan Kramosil

Measure of divergence of possibility measures 173
Tomáš Kroupa

Model defect and opportune decisions 183
Otakar Kříž

On the conditional prevision of Bruno de Finettti and its applications 187
Antonio Maturo

On maximization of the information divergence from an exponential family 199
František Matúš, Nihat Ay

Annotated graphs model for representing DAG-representable relations. Algorithmic approach and extensions 205
Azaria Paz

Nonmonotonicity and human probabilistic reasoning 221
Nikki Pfeifer, Gernot D. Kleiter

Inference in utility networks with mixed types of variables 235
Anna Slobodová

Characterization of inclusion neighbourhood in terms of the essential graph: lower neighbours 243
Milan Studený

Uncertainty and conflict: a behavioural approach to the aggregation of expert opinions 263
Matthias C. M. Troffaes

Models of logic programming with imperfect information 279
Peter Vojtáš

Noisy-or classifier 291
Jiří Vomlel

Unconstrained influence diagrams — experiments and heuristics 303
Marta Vomlelová

Logical Conditions for Coherent Qualitative and Numerical Probability Assessments

Veronica Biazzo
Dip. di Matematica e Informatica
Università di Catania, Italy
vbiazzo@dmi.unict.it

Angelo Gilio
Dip. di Metodi e Modelli Matematici
Università "La Sapienza", Roma, Italy
gilio@dmmm.uniroma1.it

Giuseppe Sanfilippo
Dip. di Matematica e Informatica
Università di Catania, Italy
gsanfilippo@dmi.unict.it

Abstract

In this paper, exploiting suitable logical conditions, we study the generalized coherence of interval-valued probability assessments and the coherence of qualitative probabilities defined on finite families of conditional events. In the numerical case, the logical conditions ensure the solvability of suitable linear systems used in the algorithm for the checking of generalized coherence. In the qualitative case, the logical conditions ensure the existence of a precise probability agreeing with the qualitative ordering.

1 Introduction

In many applications of Artificial Intelligence we need to reason with uncertain information under vague or partial knowledge. In these cases a probabilistic treatment of uncertainty based on precise probabilistic assessments is quite unrealistic. Then, a more flexible approach can be based on qualitative and/or imprecise probabilities, using suitable generalizations of the coherence principle of de Finetti, or similar principles adopted for lower and upper probabilities. Results based on such approach have been obtained in many papers (see, e.g., [1], [2], [3], [4], [5], [6], [7], [8], [9], [11], [12]).

In this paper we study lower-upper probability bounds and qualitative probabilities on finite families of conditional events. To check the consistency of conditional probability bounds we adopt a notion of generalized coherence (*g-coherence*), which is based on the coherence principle of de Finetti and is equivalent to the property of "avoiding uniform loss" given in [14]. We examine some logical conditions which allow to reduce the checking of g-coherence of upper and interval-valued conditional probability assessments to suitable subfamilies of the initial family of conditional events. Such logical conditions ensure the solvability of suitable linear systems used in the algorithm for checking g-coherence. We also consider the case of qualitative probabilities and we obtain some theoretical results which allow to represent the qualitative assessments by means of coherent precise probabilities, when some logical conditions are satisfied. We illustrate the theoretical results by some examples. Notice that similar results have been obtained in [3], [4], [5], [6], [9], [10]. The paper is organized as follows. In Section 2 we recall some preliminary notions and results. In Section 3 we give some results on the g-coherence of upper and interval-valued conditional probability bounds. In Section 4 we give some results on the coherence of qualitative probability assessments. Finally, in Section 5 we give some conclusions.

2 Preliminaries

We recall some notions and results on the coherence of precise and imprecise probability assessments. For each integer n, we set $J_n = \{1, 2, \ldots, n\}$. Given a precise probability assessment $\mathcal{P}_n = (p_j, j \in J_n)$ on a family of conditional events $\mathcal{F}_n = \{E_j|H_j, j \in J_n\}$, let $C_1, \ldots, C_m$ be the constituents, contained in $\mathcal{H}_n = \bigvee_{j=1}^{n} H_j$, which are obtained by expanding the expression

$$(E_1H_1 \vee E_1^cH_1 \vee H_1^c) \wedge \cdots \wedge (E_nH_n \vee E_n^cH_n \vee H_n^c) \,. \tag{1}$$

Let $\mathcal{S}$ be the following system

$$\left\{ \begin{array}{lll} \sum_{r\,:\,C_r \subseteq E_iH_i} \lambda_r & = & p_i \sum_{r\,:\,C_r \subseteq H_i} \lambda_r \,, \quad i \in J_n, \\ \sum_{r \in J_m} \lambda_r = 1, & & \lambda_r \geq 0, \qquad\qquad r \in J_m. \end{array} \right. \tag{2}$$

We denote respectively by $\Lambda = (\lambda_r,\, r \in J_m)$ and S the vector of unknowns and the set of solutions of the system (2) and, for each $j \in J_n$, we define $\Phi_j(\Lambda) = \sum_{r\,:\,C_r \subseteq H_j} \lambda_r$. Moreover, we define

$$I_0 = \{j \in J_n : Max_{\Lambda \in \mathcal{S}} \Phi_j(\Lambda) = 0\}. \tag{3}$$

Notice that I_0 is a (strict) subset of J_n and coincides with the set of subscripts such that, for each $j \in I_0$, the conditioning event H_j has 0 probability. Denoting by $\mathcal{P}_0$ the sub-assessment associated with the set I_0, we have

Theorem 1. The assessment $\mathcal{P}_n$ on $\mathcal{F}_n$ is coherent if and only if the following conditions are verified:
1. The system (2) is solvable; 2. if $I_0 \neq \emptyset$, then $\mathcal{P}_0$ is coherent.

Given an interval-valued probability assessment $X_n = ([l_i, u_i],\ i \in J_n)$ on a family $\mathcal{F}_n$, we use the following definition of generalized coherence (g-coherence) ([1], [12]).

Definition 1. An interval-valued probability assessment $X_n = ([l_i, u_i], i \in J_n)$, defined on a family of n conditional events $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$, is g-coherent if there exists a coherent precise probability assessment $\mathcal{P}_n = (p_i, i \in J_n)$ on $\mathcal{F}_n$, with $p_i = P(E_i|H_i)$, which is consistent with X_n, that is such that $l_i \leq p_i \leq u_i$ for each $i \in J_n$.

Generalizing the system (2) to the case of interval-valued assessments, we obtain the following system $\mathcal{S}$

$$\left\{ \begin{array}{l} \sum_{r\,:\,C_r \subseteq E_i H_i} \lambda_r \;\geq\; l_i \sum_{r\,:\,C_r \subseteq H_i} \lambda_r\,, \quad i \in J_n, \\ \sum_{r\,:\,C_r \subseteq E_i H_i} \lambda_r \;\leq\; u_i \sum_{r\,:\,C_r \subseteq H_i} \lambda_r\,, \quad i \in J_n, \\ \sum_{r \in J_m} \lambda_r = 1, \qquad \lambda_r \geq 0, \qquad r \in J_m. \end{array} \right. \tag{4}$$

Then, defining the set I_0 as in (3), Theorem 1 can be generalized to the case of interval-valued assessments in the following way.

Theorem 2. The assessment X_n on $\mathcal{F}_n$ is g-coherent if and only if the following conditions are verified:
1. The system (4) is solvable; 2. if $I_0 \neq \emptyset$, then X_0 is g-coherent.

Thus, in order to check the g-coherence of X_n we have to study the solvability of the system $\mathcal{S}$. If such system is not solvable, then X_n is not g-coherent; otherwise, we have to compute the set I_0. We denote by $(\mathcal{F}_0, X_0)$ the pair associated with I_0 and we observe that $\mathcal{F}_0$ is a strict sub-family of $\mathcal{F}_n$ and X_0 is a strict sub-vector of X_n. Then, we replace the pair $(\mathcal{F}_n, X_n)$ by $(\mathcal{F}_0, X_0)$ and we check the solvability of the (new) system $\mathcal{S}$. By repeating a finite number of times such steps, one of the following conditions is verified: (i) $\mathcal{S}$ is not solvable, which means that X_n is not g-coherent; (ii) $\mathcal{S}$ is solvable and $I_0 = \emptyset$, which means that X_n is g-coherent.

2.1 Qualitative conditional assessments

Given a permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$, we denote by $\mathcal{O}_n$ the following ordering or qualitative probability, on the family $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$,

$$P(E_{i_1}|H_{i_1}) \leq P(E_{i_2}|H_{i_2}) \leq \cdots \leq P(E_{i_n}|H_{i_n})\,, \quad E_{i_k}|H_{i_k} \in \mathcal{F}_n\,. \tag{5}$$

Definition 2. A qualitative assessment $\mathcal{O}_n$ on a family $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$ is coherent if there exists a coherent precise assessment $\mathcal{P}_n = (p_i, i \in J_n)$ on $\mathcal{F}_n$ agreeing with $\mathcal{O}_n$, that is such that $p_{i_1} \leq p_{i_2} \leq \cdots \leq p_{i_n}$.

If $\mathcal{P}_n$ is agreeing with $\mathcal{O}_n$, we also say that $\mathcal{P}_n$ *represents* the ordering $\mathcal{O}_n$.

Remark 1. From Definition 2, it follows that the qualitative assessment $\mathcal{O}_n$ is coherent if and only if there exists a g-coherent interval-valued assessment $X_n = ([l_i, u_i],\ i \in J_n)$ on $\mathcal{F}_n$, such that

$$u_{i_j} \leq l_{i_{j+1}}\,, \ \ j \in J_{n-1}\,.$$

2.2 Lower bounds

Let us consider the case in which a vector of lower bounds $\mathcal{A}_n = (l_i, i \in J_n)$ is defined on $\mathcal{F}_n$. For each constituent $C_r, r \in J_m$, we introduce a vector $V_r = (v_{r1}, \ldots, v_{rn})$, where for each $i \in J_n$ it is respectively $v_{ri} = 1$, or $v_{ri} = 0$, or $v_{ri} = l_i$, according to whether $C_r \subseteq E_i H_i$, or $C_r \subseteq E_i^c H_i$, or $C_r \subseteq H_i^c$.
Let $\mathcal{V} = \{V_r, r \in J_m\}$ be the set of vectors associated with the set of constituents $\mathcal{C} = \{C_r, r \in J_m\}$. With each $V_r \in \mathcal{V}$, $r \in J_m$, we associate the sets

$$M_r = \{i \in J_n \,:\, v_{ri} = 0\}\,, \qquad N_r = \{i \in J_n : C_r \subseteq H_i^c\}\,. \tag{6}$$

Of course, $M_r \subseteq J_n$, while $N_r \subset J_n$. Then, introducing the set

$$\mathcal{I} = \{(h,k) \,:\, h = 0, \ldots, n-1;\; k = 1, \ldots, n;\; h + k \leq n\}\,, \tag{7}$$

for each $(h,k) \in \mathcal{I}$ we define

$$\mathcal{U}_{h,k} = \{V_r \in \mathcal{V} \,:\, |N_r| = h\,,\; |M_r| = k\}\,. \tag{8}$$

Moreover, we define

$$\mathcal{W} = \{V_r \in \mathcal{V} : M_r = \emptyset\} \tag{9}$$

and

$$\mathcal{V}_h = \{V_r \in \mathcal{W} : |N_r| = h\},\; h = 0, 1, \ldots, n-1\;. \tag{10}$$

We observe that, if the sets $\mathcal{U}_{h,0}$ were defined, then we would have $\mathcal{V}_h = \mathcal{U}_{h,0}$.
We have

$$\mathcal{V} = \mathcal{W} \cup (\bigcup_{(h,k)\in\mathcal{I}} \mathcal{U}_{h,k}) = (\bigcup_{h=0}^{n-1} \mathcal{V}_h) \cup (\bigcup_{(h,k)\in\mathcal{I}} \mathcal{U}_{h,k})\,. \tag{11}$$

Given a vector of lower bounds $\mathcal{A}_n = (l_i, i \in J_n)$ on $\mathcal{F}_n = \{E_i|H_i,\; i \in J_n\}$, with $\mathcal{A}_n$ we associate the random gain $G_n = \sum_{i \in J_n} s_i H_i (E_i - l_i)$, with $s_i \geq 0$, $\forall\, i \in J_n$. We denote by $\mathcal{G} = \{g_j\}_{j \in J_m}$ the set of possible values of $G_n | \mathcal{H}_n$.

Definition 3. A set $\mathcal{T} \subset J_m$ is said a *basic set* if the following property holds: *Basic Property.* For every $r \in J_m \setminus \mathcal{T}$ there exists a set $T_r \subseteq \mathcal{T}$ such that the following condition is satisfied.

$$Max\;\{g_j\}_{j \in T_r}\; < 0 \implies g_r\; < 0\;. \tag{12}$$

Notice that the basic sets are useful to reduce the number of unknowns in the linear systems used in the checking of g-coherence ([2], [3]).
We recall some results obtained in [4], which are exploited in next section where analogous results are given for upper probability bounds.

Theorem 3. If $\mathcal{V}_0 = \mathcal{V}_1 = \cdots = \mathcal{V}_{n-1} = \emptyset$ and $l_1 + \cdots + l_n > n - 1$, then $\mathcal{A}_n$ is not g-coherent.

Theorem 4. If $\mathcal{V}_0 = \cdots = \mathcal{V}_{n-1} = \emptyset,\; |\mathcal{U}_{0,1}| = n,\; 0 < l_i < 1\; \forall\, i,$ then one has:
a) there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| = n$;
b) $\mathcal{A}_n$ is g-coherent iff $l_1 + \cdots + l_n \leq n - 1$.

Defining

$$\mathcal{Z} = \{(h,k) \,:\, h+k = n-1 \,,\; h>0\} \cup \{(h,k) \,:\, h+k < n-1\}\,,$$

we have

Theorem 5. If $\mathcal{V}_0 = \cdots = \mathcal{V}_{n-1} = \emptyset$, $\mathcal{U}_{h,k} = \emptyset$ for each $(h,k) \in \mathcal{Z}$, and $l_1 + \cdots + l_n > 1$, then $\mathcal{A}_n$ is not g-coherent.

Theorem 6. If $\mathcal{V}_0 = \cdots = \mathcal{V}_{n-1} = \emptyset$, $\mathcal{U}_{h,k} = \emptyset$, for each pair $(h,k) \in \mathcal{Z}$, $|\mathcal{U}_{0,n-1}| = n, 0 < l_i < 1 \;\forall\, i,$ then one has:
a) if, for every $j \in J_n$, it is $\sum_{i \in J_n \setminus \{j\}} l_i \leq 1$, then $\mathcal{T} = J_n$ is a basic set;
b) $\mathcal{A}_n$ is g-coherent iff $l_1 + \cdots + l_n \leq 1$.

3 Some results on conditional probability bounds

In this section we give some results on the g-coherence of upper and interval-valued probability assessments.

3.1 Upper probability assessments

Let $\mathcal{B}_n = (u_i, i \in J_n)$ be a vector of upper probability bounds defined on $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$. For each constituent $C_r, r \in J_m$, we introduce a vector $W_r = (w_{r1}, \ldots, w_{rn})$, where for each $i \in J_n$ it is respectively $w_{ri} = 1$, or $w_{ri} = 0$, or $w_{ri} = u_i$, according to whether $C_r \subseteq E_iH_i$, or $C_r \subseteq E_i^cH_i$, or $C_r \subseteq H_i^c$. Given the pair $(\mathcal{F}_n, \mathcal{B}_n)$, we construct the set $\mathcal{V}^{\mathcal{B}} = \{W_r,\, r \in J_m\}$ of the vectors associated with the set of constituents $\mathcal{C} = \{C_r,\, r \in J_m\}$. Then, with each $W_r \in \mathcal{V}^{\mathcal{B}}$ we associate the sets

$$M_r^{\mathcal{B}} = \{i \in J_n \,:\, w_{ri} = 1\}\,, \qquad N_r^{\mathcal{B}} = \{i \in J_n : C_r \subseteq H_i^c\}\,. \tag{13}$$

Of course, $M_r^{\mathcal{B}} \subseteq J_n$, while $N_r^{\mathcal{B}} \subset J_n$. Then, recalling (7), for each $(h,k) \in \mathcal{I}$ we define

$$\mathcal{U}_{h,k}^{\mathcal{B}} = \{W_r \in \mathcal{V}^{\mathcal{B}} \,:\, |N_r^{\mathcal{B}}| = h\,,\; |M_r^{\mathcal{B}}| = k\}\,. \tag{14}$$

Notice that $N_r^{\mathcal{B}} = N_r$ and $|M_r^{\mathcal{B}}| + |M_r| = n - |N_r|$. Then: $\mathcal{U}_{h,k}^{\mathcal{B}} = \mathcal{U}_{h,n-h-k}$. We define

$$\mathcal{W}^{\mathcal{B}} = \{W_r \in \mathcal{V}^{\mathcal{B}} : M_r^{\mathcal{B}} = \emptyset\} \tag{15}$$

and

$$\mathcal{V}_h^{\mathcal{B}} = \{W_r \in \mathcal{W}^{\mathcal{B}} : |N_r^{\mathcal{B}}| = h\},\; h = 0, 1, \ldots, n-1\,. \tag{16}$$

We observe that, if the sets $\mathcal{U}_{h,0}^{\mathcal{B}}$ were defined, then we would have $\mathcal{V}_h^{\mathcal{B}} = \mathcal{U}_{h,0}^{\mathcal{B}}$. We have

$$\mathcal{V}^{\mathcal{B}} = \mathcal{W}^{\mathcal{B}} \cup (\bigcup_{(h,k)\in\mathcal{I}} \mathcal{U}_{h,k}^{\mathcal{B}}) = (\bigcup_{h=0}^{n-1} \mathcal{V}_h^{\mathcal{B}}) \cup (\bigcup_{(h,k)\in\mathcal{I}} \mathcal{U}_{h,k}^{\mathcal{B}})\,. \tag{17}$$

Notice that the assessment $P(E|H) \leq u$ amounts to $P(E^c|H) \geq 1-u$. Then, a vector of upper probability bounds $\mathcal{B}_n = (u_i, i \in J_n)$ on $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$ is equivalent to the vector of lower bounds $\mathcal{A}_n = \mathcal{B}_n^c$, where $\mathcal{B}_n^c = (1-u_i, i \in J_n)$, on $\mathcal{F}_n^c = \{E_1^c|H_i, i \in J_n\}$. In other words, the g-coherence checking problems associated, respectively, with the pairs $(\mathcal{F}_n, \mathcal{B}_n)$ and $(\mathcal{F}_n^c, \mathcal{B}_n^c)$ coincide. Then, given a vector of upper bounds $\mathcal{B}_n$ on $\mathcal{F}_n$, we have

Theorem 7. If $\mathcal{V}_0^{\mathcal{B}} = \mathcal{V}_1^{\mathcal{B}} = \cdots = \mathcal{V}_{n-1}^{\mathcal{B}} = \emptyset$ and $u_1 + \cdots + u_n < 1$, then $\mathcal{B}_n$ is not g-coherent.

Proof. We replace the pair $(\mathcal{F}_n, \mathcal{B}_n)$ by the pair $(\mathcal{F}_n^c, \mathcal{B}_n^c)$. Hence, to each vector $W_r = (w_{r1}, \ldots, w_{rn})$ there corresponds, for the pair $(\mathcal{F}_n^c, \mathcal{B}_n^c)$, a vector $V_r = (v_{r1}, \ldots, v_{rn}) = (1-w_{r1}, \ldots, 1-w_{rn})$. The conditions

$$\mathcal{V}_0^{\mathcal{B}} = \mathcal{V}_1^{\mathcal{B}} = \cdots = \mathcal{V}_{n-1}^{\mathcal{B}} = \emptyset\,, \quad u_1 + \cdots + u_n < 1\,,$$

relative to $(\mathcal{F}_n, \mathcal{B}_n)$, become

$$\mathcal{V}_0 = \mathcal{V}_1 = \cdots = \mathcal{V}_{n-1} = \emptyset\,, \quad 1-u_1 + \cdots + 1-u_n > n-1\,,$$

for the pair $(\mathcal{F}_n^c, \mathcal{B}_n^c)$. Then, by Theorem 3, the vector of lower bounds $\mathcal{B}_n^c$ on $\mathcal{F}_n^c$, i.e. the vector of upper bounds $\mathcal{B}_n$ on $\mathcal{F}_n$, is not g-coherent. □

By a similar reasoning we obtain the following results.

Theorem 8. If $\mathcal{V}_0^{\mathcal{B}} = \cdots = \mathcal{V}_{n-1}^{\mathcal{B}} = \emptyset$, $|\mathcal{U}_{0,1}^{\mathcal{B}}| = n$, $0 < u_i < 1\ \forall i$, then one has:
a) there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| = n$;
b) $\mathcal{B}_n$ is g-coherent iff $u_1 + \cdots + u_n \geq 1$.

Theorem 9. If $\mathcal{V}_0^{\mathcal{B}} = \cdots = \mathcal{V}_{n-1}^{\mathcal{B}} = \emptyset$, $\mathcal{U}_{h,k}^{\mathcal{B}} = \emptyset$ for each $(h,k) \in \mathcal{Z}$, and $u_1 + \cdots + u_n < n-1$, then $\mathcal{B}_n$ is not g-coherent.

Theorem 10. If $\mathcal{V}_0 = \cdots = \mathcal{V}_{n-1} = \emptyset$, $\mathcal{U}_{h,k} = \emptyset$, for each pair $(h,k) \in \mathcal{Z}$, $|\mathcal{U}_{0,n-1}| = n, 0 < \alpha_i < 1\ \forall i$, then one has:
a) if, for every $j \in J_n$, it is $\sum_{i \in J_n \setminus \{j\}} \alpha_i \leq 1$, then $\mathcal{T} = J_n$ is a basic set;
b) $\mathcal{A}_n$ is g-coherent iff $\alpha_1 + \cdots + \alpha_n \leq 1$.

3.2 Interval-valued probability assessments

Let us consider an interval-valued probability assessments $X_n = ([l_i, u_i],\ i \in J_n)$ on a family $\mathcal{F}_n = \{E_i|H_i,\ i \in J_n\}$. We recall that the constituents $C_1, \ldots, C_m$ are contained in $\mathcal{H}_n = \bigvee_{j=1}^n H_j$. We have

Theorem 11. Given a non empty subset $\Gamma_k \subseteq J_n$, assume that the following conditions are satisfied

1. there exist two constituents C_r and C_s such that $C_r \subseteq E_i H_i$, $\forall\, i \in \Gamma_k$, $C_r \subseteq H_i^c$, $\forall\, i \in J_n \setminus \Gamma_k$, and $C_s \subseteq E_i^c H_i$, $\forall\, i \in \Gamma_k$, $C_s \subseteq H_i^c$, $\forall\, i \in J_n \setminus \Gamma_k$;
2. $Max\{l_i, i \in \Gamma_k\} \leq Min\{u_i, i \in \Gamma_k\}$.

Then the assessment $X_n = ([l_i, u_i],\ i \in J_n)$ on the family $\mathcal{F}_n$ is g-coherent iff the assessment $([l_j, u_j], j \in J_n \setminus \Gamma_k)$ on the family $\mathcal{F}_n \setminus \{E_j|H_j, j \in \Gamma_k\}$ is g-coherent.

Proof. In fact, the vector $\Lambda = (\lambda_1, \ldots, \lambda_m)$, with

$$Max\{l_i, i \in \Gamma_k\} \leq \lambda_r \leq Min\{u_i, i \in \Gamma_k\}\,,\quad \lambda_s = 1-\lambda_r\,,\quad \lambda_j = 0\,\forall\, j \in J_n \backslash \{r, s\}\,,$$

is a solution of the system (4), with $I_0 \subseteq J_n \setminus \Gamma_k$. Then, the proof follows by Theorem 2. □

In particular, observing that $l_i \leq u_i$, by the previous theorem we obtain

Corollary 1. Assume that there exist two constituents C_r, C_s such that $C_r \subseteq E_iH_i$, $C_r \subseteq H_j^c, \forall\, j \neq i$, and $C_s \subseteq E_i^cH_i$, $C_s \subseteq H_j^c, \forall\, j \neq i$. Then, the assessment $X_n = ([l_i, u_i],\ i \in J_n)$ on the family $\mathcal{F}_n$ is g-coherent iff the assessment $([l_j, u_j],\ j \in J_n \setminus \{i\})$ on the family $\mathcal{F}_n \setminus \{E_i|H_i\}$ is g-coherent.

Theorem 12. Given a subset $\Gamma_k = \{r_1, \ldots, r_k\} \subseteq J_n$, assume that the following conditions are satisfied

1. there exist k constituents $C_1, \ldots, C_k$ such that, for each $i \in J_k$, it is $C_i \subseteq E_{r_i}H_{r_i}$, $C_i \subseteq E_j^cH_j$, $\forall\, j \in \Gamma_k \setminus \{r_i\}$, and $C_i \subseteq H_j^c$, $\forall\, j \in J_n \setminus \Gamma_k$;
2. $\sum_{i\in\Gamma_k} l_i \leq 1$, $\sum_{i\in\Gamma_k} u_i \geq 1$.

Then, the assessment $X_n = ([l_i, u_i],\ i \in J_n)$ on the family $\mathcal{F}_n$ is g-coherent iff the assessment $([l_j, u_j], j \in J_n \setminus \Gamma_k)$ on the family $\mathcal{F}_n \setminus \{E_j|H_j, j \in \Gamma_k\}$ is g-coherent.

Proof. Defining the quantities $L_k = \sum_{i\in\Gamma_k} l_i\,,\quad U_k = \sum_{i\in\Gamma_k} u_i\,,$ we distinguish two cases:
(i) $L_k = U_k$. In this case, it is $l_i = u_i, \forall i \in \Gamma_k$, and the vector $\Lambda = (\lambda_1, \ldots, \lambda_m)$, with

$$\lambda_i = l_i\,,\ \forall\, i \in \Gamma_k\,,\quad \lambda_i = 0\,,\ \forall\, i \in J_n \setminus \Gamma_k\,,$$

is a solution of the system (4), with $I_0 \subseteq J_n \setminus \Gamma_k$.
(ii) $U_k > L_k$. In this case, the vector $\Lambda = (\lambda_1, \ldots, \lambda_m)$, with

$$\lambda_i = \frac{1-L_k}{U_k-L_k}u_i + \frac{U_k-1}{U_k-L_k}l_i\,,\ \ \forall\, i \in \Gamma_k\,,\quad \lambda_i = 0\,,\ \forall\, i \in J_n \setminus \Gamma_k\,,$$

is such that $l_i \leq \lambda_i \leq u_i$, $\forall\, i \in \Gamma_k$, $\quad \sum_{i\in\Gamma_k} \lambda_i = 1$. Therefore, Λ is a solution of the system (4), with $I_0 \subseteq J_n \setminus \Gamma_k$.
In both cases, the proof follows by Theorem 2. □

4 Some results on qualitative assessments

In this section we give some results on the coherence of qualitative assessments. We start by considering a family of three conditional events. We show that, in such case, under suitable logical conditions every qualitative ordering is coherent. We recall that the relation of inclusion among conditional events ([13]) is defined by

$$A|H \subseteq B|K \iff AH \subseteq BK\,,\ \ B^cK \subseteq A^cH\,. \tag{18}$$

Theorem 13. Given a family $\mathcal{F}_3 = \{E_1|H_1, E_2|H_2, E_3|H_3\}$, the ordering $\mathcal{O}_3$ defined as $P(E_1|H_1) \leq P(E_2|H_2) \leq P(E_3|H_3)$ is coherent if and only if

$$E_2|H_2 \nsubseteq E_1|H_1\,,\quad E_3|H_3 \nsubseteq E_2|H_2\,,\quad E_3|H_3 \nsubseteq E_1|H_1\,.$$

The proof of the theorem is based on the following observations. By (18), as $E_2|H_2 \nsubseteq E_1|H_1$, we have $E_2H_2 \wedge (E_1H_1)^c \neq \emptyset$, or $E_1^cH_1 \wedge (E_2^cH_2)^c \neq \emptyset$, that is: $E_1^cH_1E_2H_2 \vee H_1^cE_2H_2 \neq \emptyset$, or $E_1^cH_1E_2H_2 \vee E_1^cH_1H_2^c \neq \emptyset$.
Hence:

$$E_1^cH_1E_2H_2 \neq \emptyset\,,\ \text{ or }\ H_1^cE_2H_2 \neq \emptyset\,,\ \text{ or }\ E_1^cH_1H_2^c \neq \emptyset\,. \tag{19}$$

From $E_3|H_3 \nsubseteq E_2|H_2$ and $E_3|H_3 \nsubseteq E_1|H_1$ we obtain, respectively

$$E_2^cH_2E_3H_3 \neq \emptyset\,,\ \text{ or }\ H_2^cE_3H_3 \neq \emptyset\,,\ \text{ or }\ E_2^cH_2H_3^c \neq \emptyset\,. \tag{20}$$

$$E_1^cH_1E_3H_3 \neq \emptyset\,,\ \text{ or }\ H_1^cE_3H_3 \neq \emptyset\,,\ \text{ or }\ E_1^cH_1H_3^c \neq \emptyset\,. \tag{21}$$

By (19), (20), (21), it can be proved that there exists a coherent precise assessment $\mathcal{P}_3 = (p_1, p_2, p_3)$ on $\mathcal{F}_3$, such that $p_1 \leq p_2 \leq p_3$, so that $\mathcal{O}_3$ is coherent.
Considering the case of n conditional events, we examine some logical conditions which ensure that every qualitative ordering $\mathcal{O}_n$ is coherent.

Theorem 14. Given a family $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$, assume that there exist n constituents $C_1, \ldots, C_n$ such that, for each $r \in J_n$, it is $C_r \subseteq E_r^cH_r, C_r \subseteq E_jH_j, \forall\, j \in J_n \setminus \{r\}$. Then, for each permutation $(i_1, \ldots, i_n)$, the ordering $\mathcal{O}_n$ defined by (5) is coherent.

Proof. We have to prove that, for each qualitative assessment $P(E_{i_1}|H_{i_1}) \leq P(E_{i_2}|H_{i_2}) \leq \cdots \leq P(E_{i_n}|H_{i_n})$, there exists a coherent precise probability assessment $(p_1, \ldots, p_n)$ on $\mathcal{F}_n$, with $p_i = P(E_i|H_i)$, agreeing with such ordering, i.e. such that $p_{i_1} \leq p_{i_2} \leq \cdots \leq p_{i_n}$. For each $r \in J_n$, with the constituent C_r we associate the variable λ_r. We first consider the ordering $P(E_1|H_1) \leq P(E_2|H_2) \leq \cdots \leq P(E_n|H_n)$. Given n quantities $q_1, \ldots, q_n$, with $q_j \geq 0$, $\sum_{j=1}^n q_j = 1$ and $q_n \leq q_{n-1} \leq \cdots \leq q_1$, let us consider the precise assessment $\mathcal{P}_n = (p_1, \ldots, p_n)$ on $\mathcal{F}_n$ defined as $p_i = \sum_{j \neq i} q_j = 1 - q_i$. Of course, $p_1 \leq p_2 \leq \cdots \leq p_n$. Moreover, the vector $\Lambda = (\lambda_1, \ldots, \lambda_m)$, with $\lambda_i = q_i, \forall\, i \in J_n$, $\lambda_i = 0, \forall\, i \in J_m \setminus J_n$, is a solution of the system (2) associated with the pair $(\mathcal{F}_n, \mathcal{P}_n)$, with $I_0 = \emptyset$. Therefore, $\mathcal{P}_n$ is coherent and the corresponding qualitative assessment is coherent too.
In a similar way, we can prove, for each permutation $(i_1, \ldots, i_n)$, the coherence of the ordering (5). □

Theorem 15. Given a family of n conditional events $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$, assume that there exist n constituents $C_1, \ldots, C_n$ such that, for each $r \in J_n$, it is $C_r \subseteq E_r H_r, C_r \subseteq E_j^c H_j, \forall\, j \in J_n \setminus \{r\}$. Then, for each permutation $(i_1, \ldots, i_n)$, the ordering $\mathcal{O}_n$ defined by (5) is coherent.

Proof. By the hypotheses, we have that for the family $\mathcal{F}_n^c = \{E_1^c|H_1, \ldots, E_n^c|H_n\}$ there exist n constituents $C'_1, \ldots, C'_n$ such that, for each $r \in J_n$, it is $C'_r \subseteq E_r^c H_r, C'_r \subseteq E_j H_j, \forall\, j \in J_n \setminus \{r\}$. Then, by Theorem 14, for each permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$, the qualitative assessment $\mathcal{O}_n^c$ defined as

$$P(E_{i_1}^c|H_{i_1}) \leq P(E_{i_2}^c|H_{i_2}) \leq \cdots \leq P(E_{i_n}^c|H_{i_n})$$

is coherent. Of course, the ordering on $\mathcal{F}_n^c$ associated with the permutation $(i_n, \ldots, i_1)$ is coherent too. Then, the conclusion follows by observing that the coherence of such ordering is equivalent to the coherence of the qualitative assessment $\mathcal{O}_n$ on $\mathcal{F}_n$, associated with the permutation $(i_1, \ldots, i_n)$. □

Theorem 16. Given a family $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$ and a subscript $j \in J_n$, assume that there exist $n-1$ constituents C_r, $r \in J_n \setminus \{j\}$, such that $C_r \subseteq E_r^c H_r$, $C_r \subseteq H_j^c$, $C_r \subseteq E_i H_i$, $\forall\, i \in J_n \setminus \{r, j\}$. Then, for each permutation $(i_1, \ldots, i_n)$, the ordering $\mathcal{O}_n$ defined by (5) is coherent.

Proof. We have to prove that, for each ordering $P(E_{i_1}|H_{i_1}) \leq P(E_{i_2}|H_{i_2}) \leq \cdots \leq P(E_{i_n}|H_{i_n})$, there exists a coherent precise assessment $(p_1, \ldots, p_n)$ agreeing with such ordering. For each $i \in J_n \setminus \{j\}$, with the constituent C_i we associate the unknown λ_i. We first consider the ordering $P(E_1|H_1) \leq P(E_2|H_2) \leq \cdots \leq P(E_n|H_n)$. Given n quantities $q_1, \ldots, q_n$, with $q_1 \geq \cdots \geq q_n$, $q_k \geq 0$, $\forall\, k \in J_n$, $\sum_{k \in J_n \setminus \{j\}} q_k = 1$, let us consider the precise assessment $(p_1, \ldots, p_n)$ on $\mathcal{F}_n$ defined as $p_i = \sum_{k \in J_n \setminus \{i,j\}} q_k = 1 - q_i$, for $i \neq j$, and with p_j an arbitrary number such that $p_{j-1} \leq p_j \leq p_{j+1}$, where we set $p_{j-1} = 0$ if $j = 1$ and $p_{j+1} = 1$ if $j = n$. Of course, it is $p_1 \leq \cdots \leq p_n$. Moreover, the vector $\Lambda = (\lambda_1, \ldots, \lambda_m)$, with $\lambda_j = 0$, $\lambda_i = q_i, \forall\, i \in J_n \setminus \{j\}$, is a solution of the system (2), with $I_0 \subseteq I'_0 = \{j\}$. Then, as the assessment $P(E_j|H_j) = p_j$ is coherent, the assessment $(p_1, \ldots, p_n)$ is a precise coherent assessment on $\mathcal{F}_n$ agreeing with $P(E_1|H_1) \leq P(E_2|H_2) \leq \cdots \leq P(E_n|H_n)$.
In a similar way, we can prove, for each permutation $(i_1, \ldots, i_n)$, the coherence of the ordering (5). □

Theorem 17. Given a family of n conditional events $\mathcal{F}_n = \{E_1|H_1, \ldots, E_n|H_n\}$ and a subscript $j \in J_n$, assume that there exist $n-1$ constituents C_h, $h \in J_n \setminus \{j\}$, such that $C_h \subseteq E_h H_h$, $C_h \subseteq H_j^c$, $C_h \subseteq E_i^c H_i$, $\forall\, i \in J_n \setminus \{h, j\}$. Then, for each permutation $(i_1, \ldots, i_n)$, the ordering $\mathcal{O}_n$ defined by (5) is coherent.

Proof. By the hypotheses, we have that for the family $\mathcal{F}_n^c = \{E_1^c|H_1, \ldots, E_n^c|H_n\}$ there exist $n-1$ constituents C'_r, $r \in J_n \setminus \{j\}$, such that $C'_r \subseteq E_r^c H_r$, $C'_r \subseteq H_j^c$, $C'_r \subseteq E_i H_i$, $\forall\, i \in J_n \setminus \{r, j\}$. Then, by Theorem 16, for each permutation $(i_1, \ldots, i_n)$, the qualitative assessment $\mathcal{O}_n^c$ defined as

$$P(E_{i_1}^c|H_{i_1}) \leq P(E_{i_2}^c|H_{i_2}) \leq \cdots \leq P(E_{i_n}^c|H_{i_n})$$

is coherent. Of course, the ordering on $\mathcal{F}_n^c$ associated with the permutation $(i_n, \dots, i_1)$ is coherent too. Then, the conclusion follows by observing that the coherence of such ordering is equivalent to the coherence of the qualitative assessment $\mathcal{O}_n$ on $\mathcal{F}_n$, associated with the permutation $(i_1, \dots, i_n)$. □

Example 1. We will now examine an example which is an application of Theorem 17. Given the family $\mathcal{F}_3 = \{E_1|H_1, E_2|H_2, E_3|H_3\} = \{A|B,\ D|B,\ AB|C\}$, where the events A, B, C, D are logically independent, let us consider the qualitative assessment $P(E_1|H_1) \leq P(E_2|H_2) \leq P(E_3|H_3)$. We define $C_1 = ABC^cD^c$, $C_2 = A^cBC^cD$ and with C_1 and C_2 we associate, respectively, the unknowns λ_1 and λ_2. Moreover, we set $\lambda_r = 0, \forall\, r > 2$. Given three quantities q_1, q_2, q_3, with $q_1 \leq q_2 \leq q_3$ and $q_1 + q_2 = 1$, let us consider the precise assessment $(p_1, p_2, p_3) = (q_1, q_2, q_3)$ on the family $\mathcal{F}_3$. In our case, the system (2) becomes

$$\begin{cases} \lambda_1 = p_1(\lambda_1 + \lambda_2), \\ \lambda_2 = p_2(\lambda_1 + \lambda_2), \\ 0 = p_3 \cdot 0, \\ \lambda_1 + \lambda_2 = 1, \quad \lambda_r \geq 0,\ r = 1, 2, \end{cases} \tag{22}$$

and has the solution $\lambda_1 = q_1$, $\lambda_2 = q_2$, with $I_0 \subseteq \{3\}$. Then, as the assessment $P(AB|C) = p_3$ is coherent, (p_1, p_2, p_3) is a precise coherent assessment on $\mathcal{F}_3$ agreeing with the qualitative assessment $P(E_1|H_1) \leq P(E_2|H_2) \leq P(E_3|H_3)$.

Theorem 18. Given a family $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$, assume that there exist n constituents $C_1, \dots, C_n$, with

$$C_i \subseteq E_iH_iE_{i+1}H_{i+1}\,,\ \forall\, i \in J_{n-1}\,, \quad C_i \subseteq E_j^cH_j\,,\ \forall\, j \in J_n \setminus \{i, i+1\}\,,$$

and $C_n \subseteq E_nH_n\,,\ C_n \subseteq E_j^cH_j\,,\ \forall\, j \in J_{n-1}$. Then, for every $k \in J_n \setminus \{1\}$, the ordering $\mathcal{O}_n$ on $\mathcal{F}_n$, associated with the permutation $(1, k, \dots, n, 2, \dots, k-1)$, is coherent.

Proof. We will prove the theorem by showing that there exists a precise assessment $\mathcal{P}_n = (p_1, \dots, p_n)$ on $\mathcal{F}_n$, with $p_1 \leq p_k \leq \cdots \leq p_n \leq p_2 \leq \cdots \leq p_{k-1}$, such that the system (2) has a solution with $I_0 = \emptyset$, so that $\mathcal{P}_n$ is coherent and represents the ordering $\mathcal{O}_n$ associated with the permutation $(1, k, \dots, n, 2, \dots, k-1)$. For each $i \in J_n$, with the constituent C_i we associate the unknown λ_i. Moreover, we set $\lambda_r = 0, \forall\, r > n$. The system (2), with vector of unknowns $\Lambda = (\lambda_1, \dots, \lambda_n, 0, \dots, 0)$, is the following one

$$p_1 = \lambda_1\,,\ p_i = \lambda_{i-1} + \lambda_i\,,\ i = 2, \dots, n\,;\ \sum_{r \in J_n} \lambda_r = 1\,;\ \lambda_r \geq 0,\ \forall\, r \in J_n\,.$$

Then, in order the inequalities $p_1 \leq p_k \leq \cdots \leq p_n \leq p_2 \leq \cdots \leq p_{k-1}$ be satisfied, the following system must be solvable

$$\begin{cases} \lambda_1 \leq \lambda_{k-1} + \lambda_k \leq \cdots \leq \lambda_{n-1} + \lambda_n \leq \lambda_1 + \lambda_2 \leq \cdots \leq \lambda_{k-2} + \lambda_{k-1}\,, \\ \\ \sum_{r \in J_n} \lambda_r = 1,\ \lambda_r \geq 0,\ r \in J_n\,. \end{cases} \tag{23}$$

We should distinguish different cases. As an example, let us assume that n is an even number and k is an odd number. Then, it must be

$$\lambda_1 \leq \lambda_{k-1} + \lambda_k\,; \quad \lambda_{n-1} + \lambda_n \leq \lambda_1 + \lambda_2\,; \quad \lambda_2 \leq \lambda_4 \leq \cdots \leq \lambda_{n-2} \leq \lambda_n\,;$$
$$\lambda_k \leq \lambda_{k+2} \leq \cdots \leq \lambda_{n-3} \leq \lambda_{n-1}\,; \quad \lambda_1 \leq \lambda_3 \leq \cdots \leq \lambda_{k-4} \leq \lambda_{k-2}\,;$$

In this case, a solution of the system, with $I_0 = \emptyset$, is

$$\lambda_{n-1} = \lambda_n = \tfrac{c}{5}\,, \quad \lambda_{k-1} = \lambda_k = \lambda_{k+2} = \cdots = \lambda_{n-3} = \tfrac{c}{6}\,,$$
$$\lambda_2 = \lambda_4 = \cdots = \lambda_{n-2} = \tfrac{c}{7}\,, \quad \tfrac{9c}{35} \leq \lambda_1 = \lambda_3 = \cdots = \lambda_{k-2} \leq \tfrac{c}{3}\,,$$

where c is a suitable positive constant such that $\sum_{r=1}^{n} \lambda_r = 1$.
By a similar reasoning we can examine the other cases. □

Notice that Theorem (18) holds also for a "strict" ordering (where each inequality $\leq$ is replaced by $<$). But, Theorem (18) doesn't hold when, in the strict ordering, the term $P(E_1|H_1)$ is not the first one), as shown in the next example.

Example 2. Let $F = \{E_1|H_1, E_2|H_2, E_3|H_3\}$ be a family of three conditional events such that there exist the constituents

$$C_1 = E_1H_1E_2H_2E_3^cH_3\,, \quad C_2 = E_1^cH_1E_2H_2E_3H_3\,, \quad C_3 = E_1^cH_1E_2^cH_2E_3H_3\,.$$

Consider the ordering $P(E_3|H_3) \leq P(E_2|H_2) \leq P(E_1|H_1)$. The associated system is

$$\begin{cases} \lambda_2 + \lambda_3 \leq \lambda_1 + \lambda_2, \\ \lambda_1 + \lambda_2 \leq \lambda_1\,, \\ \lambda_1 + \lambda_2 + \lambda_3 = 1\,, \quad \lambda_r \geq 0\,, \, r = 1, 2, 3. \end{cases}$$

Each solution $(\lambda_1, \lambda_2, \lambda_3)$ of the system is such that $\lambda_2 = 0$, $\lambda_3 \leq \lambda_1$, and hence it must be $p_3 = \lambda_3 \leq p_1 = p_2 = \lambda_1$, so that the ordering $P(E_3|H_3) < P(E_2|H_2) < P(E_1|H_1)$ is not coherent.

5 Conclusions

In this paper we have considered some qualitative and numerical, upper or interval-valued, conditional probability assessments, defined on finite families of conditional events. In the numerical case, we have examined some logical conditions which ensure the solvability of suitable linear systems. Such linear systems are used in the algorithm which check the g-coherence of the conditional probability bounds. In the qualitative case, we have examined some logical conditions which are sufficient for the existence of a precise probability agreeing with the qualitative ordering on the given conditional events.

References

[1] Biazzo V., and Gilio A. (2000), A generalization of the fundamental theorem of de Finetti for imprecise conditional probability assessments, *International Journal of Approximate Reasoning 24*, 251-272.

[2] Biazzo V., Gilio A. (2002), On the linear structure of betting criterion and the checking of coherence, *Annals of Mathematics and Artificial Intelligence 35*, 83-106.

[3] Biazzo V., Gilio A., and Sanfilippo G. (2003), Coherence Checking and Propagation of Lower Probability Bounds, *Soft Computing 7*, 310-320.

[4] Biazzo V., Gilio A., and Sanfilippo G. (2003), Some results on generalized coherence of conditional probability bounds, *Proc. of The Third International Symposium on Imprecise Probabilities and their Applications (ISIPTA '03)*, Lugano, Switzerland, July 14 - 17, 62-76.

[5] Capotorti A., Galli L., and Vantaggi B. (2003), Locally strong coherence and inference with lower-upper probabilities, *Soft Computing 7*, 280-287.

[6] Capotorti A., and Vantaggi B. (2002), Locally strong coherence in inference processes, *Annals of Mathematics and Artificial Intelligence 35*, 125-149.

[7] Coletti G. (1990), Coherent qualitative probability, *Journal of Mathematical Psychology 34*, 297-310.

[8] Coletti G. (1994), Coherent numerical and ordinal probabilistic assessments, *IEEE Trans. on Systems, Man, and Cybernetics 24 (12)*, 1747-1754.

[9] Coletti G., Gilio A., and Scozzafava R. (1991), Conditional events with vague information in expert systems, in: *Uncertainty in Knowledge Bases (Bouchon-Meunier, B., Yager, R. R., Zadeh, L. A. (Eds.)), Lecture Notes in Computer Science 521*, Springer-Verlag Berlin Heidelberg, 106-114.

[10] Coletti G., Gilio A., and Scozzafava R. (1993), Comparative probability for conditional events : a new look through coherence, *Theory and Decision 35*, 237-258.

[11] Coletti G., and Scozzafava R. (1999), Coherent upper and lower Bayesian updating, *Proc. of the First Intern. Symposium on Imprecise Probabilities and Their Applications*, (Eds. de Cooman G., Cozman F.G., Moral S., Walley P.), Imprecise Probabilities Project, Universiteit Gent, Belgium, 101-110.

[12] Gilio A. (1995), Probabilistic consistency of conditional probability bounds, in: *Advances in Intelligent Computing (B. Bouchon-Meunier, R. R. Yager, and L. A. Zadeh, Eds.), Lecture Notes in Computer Science 945*, Springer-Verlag, Berlin Heidelberg, 200-209.

[13] Goodman, I. R., and H. T. Nguyen (1988), Conditional objects and the modeling of uncertainties, in: *Fuzzy Computing Theory, Hardware and Applications (M. M. Gupta, and T. Yamakawa, Eds.)*, North-Holland, New York.

[14] Walley P. (1991), *Statistical reasoning with imprecise probabilities*, Chapman and Hall, London.