On general conditional prevision assessments

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Abstract

In this paper we consider general conditional random quantities of the kind $X|Y$, where X and Y are finite discrete random quantities. Then, we introduce the notion of coherence for conditional prevision assessments on finite families of general conditional random quantities. Moreover, we give a compound prevision theorem and we examine the relation between the previsions of $X|Y$ and $Y|X$. Then, we give some results on random gains and, by a suitable alternative theorem, we obtain a characterization of coherence. We also propose an algorithm for the checking of coherence. Finally, we briefly examine the case of imprecise conditional prevision assessments by introducing the notions of generalized and total coherence. To illustrate our results, we consider some examples.

1 Introduction

In a recent paper ([1]) we have studied the notion of general conditional prevision $\mathbb{P}(X|Y)$, where X and Y are finite discrete random quantities. This general notion of conditional prevision has been introduced by Lad and Dickey in [5] and also discussed in [6]. In their work Lad and Dickey consider a notion of conditional prevision of the form $\mathbb{P}(X|Y)$ where both X and Y are random quantities, by generalizing the de Finetti's definition of a conditional prevision assertion $\mathbb{P}(X|H)$, where H is an event. In [5, 6] the case $\mathbb{P}(Y) = 0$ has not been considered; on the other hand, $\mathbb{P}(Y) = 0$ doesn't imply $\mathbb{P}(XY) = 0$; then $\mathbb{P}(X|Y)$ might not exist. In order to handle the case $\mathbb{P}(Y) = 0$ in [1] we have proposed a notion of coherence which integrates the definition of $\mathbb{P}(X|Y)$ given by Lad and Dickey. In particular, among other results, we have given a strong

generalized compound prevision theorem. In this paper we continue our study, by considering in general conditional prevision assessments on finite families of finite discrete conditional random quantities. We introduce in general the notion of coherence; we examine the compound prevision theorem and a kind of generalization of Bayes theorem; we obtain some results on random gains; moreover, we give some results to characterize coherence and, by exploiting them, we propose an algorithm for the checking of coherence; finally, we consider the case of imprecise conditional prevision assessments, by introducing the notions of generalized coherence and total coherence. We illustrate our results by some examples.
The paper is organized as follows: in Section 2 we give some preliminary notions and results; in Section 3 we introduce in general the notion of coherence for conditional prevision assessments; in Section 4 we generalize the compound prevision theorem and we examine the relation between $\mathbb{P}(X|Y)$ and $\mathbb{P}(Y|X)$; in Section 5 we give some results on random gains; in Section 6 we illustrate a procedure, by proposing an algorithm, for the checking of coherence; in Section 7 we briefly examine the case of imprecise conditional prevision assessments by introducing the notions of generalized and total coherence; finally, in Section 8 we give some conclusions and comments on future work.

2 Some preliminary notions and results

We recall below two definitions given in [5, 6].

Definition 1. The conditional prevision for X given Y, denoted $\mathbb{P}(X|Y)$, is a number you specify with the understanding that you accept to engage any transaction yielding a random net gain $G = sY[X - \mathbb{P}(X|Y)]$, where s is an arbitrary real quantity.

Definition 2. Having asserted your conditional prevision $\mathbb{P}(X|Y) = \mu$, the conditional random quantity $X|Y$ is defined as

$$X|Y = XY + (1 - Y)\mu = \mu + Y(X - \mu)\,. \tag{1}$$

In [1] some critical comments and examples have been given on the previous definitions. Then, based on the notion of coherence given in [2, 4, 7, 8, 9], the following definition has been proposed

Definition 3. Given two random quantities X, Y and a conditional prevision assessment $\mathbb{P}(X|Y) = \mu$, let $G = s(X|Y - \mu) = sY(X - \mu)$ be the net random gain, where s is an arbitrary real quantity, with $s \neq 0$. Defining the event $H = (Y \neq 0)$, the assessment $\mathbb{P}(X|Y) = \mu$ is coherent if and only if: $inf\, G|H \cdot sup\, G|H \leq 0$, for every s.

Let be $X \in \mathcal{C}_X = \{x_1, \ldots, x_n\}$ and $Y \in \mathcal{C}_Y = \{y_1, \ldots, y_r\}$, with $y_k \geq 0\,,\ \forall\, k$, and $(X, Y) \in \mathcal{C} \subseteq \mathcal{C}_X \times \mathcal{C}_Y$. We denote by X^0 the subset of $\mathcal{C}_X$ such that for each $x_h \in X^0$ there exists $(x_h, y_k) \in \mathcal{C}$ with $y_k \neq 0$. Then, we set

$$x_0 = min\, X^0\,, \quad x^0 = max\, X^0\,. \tag{2}$$

Then, we have ([1])

Theorem 1. Given two finite random quantities X, Y, with $Y \geq 0$, the prevision assessment $\mathbb{P}(X|Y) = \mu$ is coherent if and only if $x_0 \leq \mu \leq x^0$.

A similar result holds for $Y \leq 0$.

3 Coherence of general conditional previsions

Given any random quantities $X_1, \ldots, X_n, Y_1, \ldots, Y_n$, based on Definitions 1 and 2 we denote by $\mathcal{M}_n = (\mu_1, \ldots, \mu_n)$ a vector of conditional previsions for "X_1 given Y_1", ..., "X_n given Y_n", where $\mu_i = \mathbb{P}(X_i|Y_i)$; then, we set $\mathcal{F}_n = \{X_1|Y_1, \ldots, X_n|Y_n\}$ and we denote by

$$\mathcal{G}_n = \sum_i^n s_i(X_i|Y_i - \mu_i) = \sum_i^n s_i Y_i (X_i - \mu_i)\,,$$

where $s_1, \ldots, s_n$ are arbitrary real quantities, the random gain associated with the pair $(\mathcal{F}_n, \mathcal{M}_n)$. We set $H_i = (Y_i \neq 0)$, $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$; then, based on [2, 4, 7, 8, 9], we generalize Definition 3 by the following

Definition 4. Let $\mathbb{P}$ be a real function defined on a family $\mathcal{K}$ of conditional random quantities. $\mathbb{P}$ is said coherent if and only if, for every integer n, for every $s_1, \ldots, s_n$, and for every sub-family $\mathcal{F}_n \subseteq \mathcal{K}$, denoting by $\mathcal{M}_n = (\mu_1, \ldots, \mu_n)$ the restriction of $\mathbb{P}$ to $\mathcal{F}_n$, the following condition is satisfied

$$inf\ \mathcal{G}_n|\mathcal{H}_n \ \leq\ 0\ \leq\ sup\ \mathcal{G}_n|\mathcal{H}_n\,, \tag{3}$$

which is equivalent to $inf\ \mathcal{G}_n|\mathcal{H}_n \ \leq\ 0$, or $sup\ \mathcal{G}_n|\mathcal{H}_n \geq 0$.

We give below an example where, based on Definition 4, it is shown that in some cases do not exist finite coherent conditional prevision assessments.

Example 1. Let be given a random quantity $X \in \{-1, 1\}$, with $\mathbb{P}(X) = 0$, i.e. $P(X = -1) = P(X = 1) = \frac{1}{2}$. Of course, it is $X^2 = \mathbb{P}(X^2) = 1$; hence, the assessment $\mathbb{P}(X) = 0$ has the unique extension $\mathbb{P}(X^2) = 1$. It can be shown that the assessment $(0, 1)$ on $\{X, X^2\}$ has no finite extensions on $X|X$. In fact, let $\mathcal{M}_3 = (0, 1, \mu)$ be a prevision assessment on $\mathcal{F}_3 = \{X, X^2, X|X\}$, where $\mu = \mathbb{P}(X|X)$. By compound prevision theorem, $\mathbb{P}(XY) = \mathbb{P}(Y)\mathbb{P}(X|Y)$, it should be $\mathbb{P}(X^2) = \mathbb{P}(X)\mathbb{P}(X|X)$, that is: $1 = 0 \cdot \mu$, which has no finite solutions in the unknown μ. We will show that, for every finite quantity μ, the condition of coherence is not satisfied. In our case $H_1 = H_2 = H_3 = \Omega = \mathcal{H}_3$, so that

$$\mathcal{G}_3|\mathcal{H}_3 = \mathcal{G}_3 = s_1(X-0)+s_2(X^2-1)+s_3X(X-\mu) = (s_1-s_3\mu)X+(s_2+s_3)X^2-s_2\,;$$

then, denoting by g_1 (resp., g_2) the value of $\mathcal{G}_3$ associated with $X = -1$ (resp., $X = 1$), it is $g_1 = -s_1 + (1 + \mu)s_3,\ g_2 = s_1 + (1 - \mu)s_3$. Hence $s_1 < (1 + \mu)s_3$

implies $g_1 > 0$, while $s_1 > (-1+\mu)s_3 = (1+\mu)s_3 - 2s_3$ implies $g_2 > 0$. Then, for every pair (s_1, s_3), with $s_3 > 0$ and $(1+\mu)s_3 < s_1 < -2s_3(1+\mu)s_3$ it is $g_1 > 0$, $g_2 > 0$; that is: $inf\ \mathcal{G}_3|\mathcal{H}_3 > 0$. Thus, the assessment $(0, 1, \mu)$ on $\{X, X^2, X|X\}$ is not coherent, for every finite μ.
We remark that, still assuming $X \in \{-1, 1\}$ and $\mathbb{P}(X) = 0$, the incoherence of the assessment $\mathbb{P}(X|X) = \mu$ can be proved by directly observing that it should be $\mathbb{P}[(X|X) - \mu] = 0$; that is $\mathbb{P}[X(X-\mu)] = \mathbb{P}(X^2 - \mu X) = 1 - \mu \cdot 0 = 0$, which is false, for every μ.

4 Compound prevision and Bayes theorems

We give below a result which generalizes the compound probability theorem to the case of n arbitrary random quantities $X_1, \ldots, X_n$.

Theorem 2. Given n random quantities $X_1, \ldots, X_n$, we have

$$\mathbb{P}(X_1 \cdots X_n) = \mathbb{P}(X_1)\mathbb{P}(X_2|X_1) \cdots \mathbb{P}(X_n|X_1 \cdots X_{n-1})\,.$$

Proof. The proof immediately follows by the compound prevision theorem; in fact, by suitably iterating the formula $\mathbb{P}(XY) = \mathbb{P}(Y)\mathbb{P}(X|Y)$, we have

$$\mathbb{P}(X_1 \cdots X_n) = \mathbb{P}(X_1 \cdots X_{n-1})\mathbb{P}(X_n|X_1 \cdots X_{n-1}) =$$

$$= \mathbb{P}(X_1 \cdots X_{n-2})\mathbb{P}(X_{n-1}|X_1 \cdots X_{n-2})\mathbb{P}(X_n|X_1 \cdots X_{n-1}) = \cdots =$$

$$= \mathbb{P}(X_1)\mathbb{P}(X_2|X_1) \cdots \mathbb{P}(X_n|X_1 \cdots X_{n-1})\,.$$

□

The following result gives a kind of generalization of Bayes theorem, by analyzing the relationship between $\mathbb{P}(X|Y)$ and $\mathbb{P}(Y|X)$.

Theorem 3. Given two finite random quantities X, Y, with $\mathbb{P}(X) \neq 0$, we have

$$\mathbb{P}(Y|X) = \mathbb{P}(X|Y) \cdot \frac{\sum_j y_j P(Y = y_j)}{\sum_j P(Y = y_j)\mathbb{P}(X|Y = y_j)}\,.$$

Proof. We have $\mathbb{P}(XY) = \mathbb{P}(Y)\mathbb{P}(X|Y) = \mathbb{P}(X)\mathbb{P}(Y|X)$; then

$$\mathbb{P}(Y|X) = \mathbb{P}(X|Y) \cdot \frac{\mathbb{P}(Y)}{\mathbb{P}(X)} = \mathbb{P}(X|Y) \cdot \frac{\sum_j y_j P(Y = y_j)}{\sum_j P(Y = y_j)\mathbb{P}(X|Y = y_j)}\,.$$

□

Given any event E and a random quantity Y, with $\mathbb{P}(Y) \neq 0$, we have

$$\mathbb{P}(E|Y) = \frac{\mathbb{P}(Y|E)P(E)}{\mathbb{P}(Y)} = P(E) \cdot \frac{\sum_j y_j P(Y = y_j|E)}{\sum_j y_j P(Y = y_j)}\,.$$

Moreover, given two logically incompatible events A and B, we have

$$\mathbb{P}(A \vee B|Y) = \mathbb{P}(A + B|Y) = \mathbb{P}(A|Y) + \mathbb{P}(B|Y) =$$

$$= P(A) \cdot \frac{\sum_j y_j P(Y = y_j|A)}{\sum_j y_j P(Y = y_j)} + P(B) \cdot \frac{\sum_j y_j P(Y = y_j|B)}{\sum_j y_j P(Y = y_j)} \,.$$

5 Some results on random gains

In this section we deepen the notion of coherence given in Definition 4 and we obtain further theoretical results. Given any integer n, we set $J_n = \{1, \ldots, n\}$. Let be given a conditional prevision assessment $\mathcal{M}_n = (\mu_i, i \in J_n)$ on a family $\mathcal{F}_n = \{X_i|Y_i, i \in J_n\}$ of n conditional random quantities, where $\mu_i = \mathbb{P}(X_i|Y_i)$. For each subset $K \subseteq J_n$, we set $\mathcal{H}_K = \bigvee_{i \in K} H_i$; moreover, considering the sub-assessment $\mathcal{M}_K = (\mu_i, i \in K)$ on the sub-family $\mathcal{F}_K = \{X_i|Y_i, i \in K\}$, we denote by $\mathcal{G}_K$ the random gain associated with the pair $(\mathcal{F}_K, \mathcal{M}_K)$. Of course, $\mathcal{G}_n = \mathcal{G}_{J_n}$ and $\mathcal{F}_n = \mathcal{F}_{J_n}$. We denote by $\mathcal{K}$ the class of the sets $K \subseteq J_n$ which satisfy the condition $inf\, \mathcal{G}_n|\mathcal{H}_K \cdot sup\, \mathcal{G}_n|\mathcal{H}_K > 0$ for some $s_i \in \mathbb{R}, i \in J_n$. Of course, $\mathcal{K}$ may be empty. We have

Theorem 4. The class $\mathcal{K}$ is additive; that is, for every $K' \in \mathcal{K}, K'' \in \mathcal{K}$, it is $K' \cup K'' \in \mathcal{K}$. Moreover, for every $K \in \mathcal{K}$, if $K' \subset K$, then $K' \in \mathcal{K}$.

Proof. Assume that $K' \in \mathcal{K}, K'' \in \mathcal{K}$; i.e., $inf\, \mathcal{G}_n|\mathcal{H}_{K'} > 0$, $inf\, \mathcal{G}_n|\mathcal{H}_{K''} > 0$. We observe that the set of values of $\mathcal{G}_n|\mathcal{H}_{K' \cup K''}$ is the union of the set of values of $\mathcal{G}_n|\mathcal{H}_{K'}$ and $\mathcal{G}_n|\mathcal{H}_{K''}$; therefore

$$inf\, \mathcal{G}_n|\mathcal{H}_{K' \cup K''} = min\, \{inf\, \mathcal{G}_n|\mathcal{H}_{K'},\, inf\, \mathcal{G}_n|\mathcal{H}_{K''}\} > 0\,;$$

hence $K' \cup K'' \in \mathcal{K}$. Moreover, given any $K \in \mathcal{K}$ and any $K' \subset K$, as $\mathcal{H}_{K'} \subseteq \mathcal{H}_K$, the set of values of $\mathcal{G}_n|\mathcal{H}_{K'}$ is contained in the set of values of $\mathcal{G}_n|\mathcal{H}_K$ and hence $inf\, \mathcal{G}_n|\mathcal{H}_{K'} \geq inf\, \mathcal{G}_n|\mathcal{H}_K > 0$; therefore $K' \in \mathcal{K}$. □

We set

$$K_0 = \bigcup_{K \in \mathcal{K}} K\,, \quad \Gamma_0 = J_n \setminus K_0\,. \tag{4}$$

Of course, $K_0 \in \mathcal{K}$ and $\mathcal{K}$ is the power set of K_0; in conclusion, given any $K \subseteq J_n$, it is $K \setminus K_0 \neq \emptyset$, i.e. $K \notin \mathcal{K}$, if and only if $inf\, \mathcal{G}_n|\mathcal{H}_K \leq 0$. Then, we have

Theorem 5. Given a family $\mathcal{F}_n = \{X_i|Y_i, i \in J_n\}$ of n conditional random quantities and any conditional prevision $\mathcal{M}_n = (\mu_i, i \in J_n)$ on $\mathcal{F}_n$, let $(\mathcal{F}_{\Gamma_0}, \mathcal{M}_{\Gamma_0})$ be the pair associated with the subset Γ_0 defined as in (4). The conditional prevision sub-assessment $\mathcal{M}_{\Gamma_0}$ on the sub-family $\mathcal{F}_{\Gamma_0}$ is coherent.

Proof. Based on Definition 4, we have to prove that, for every $J \subseteq \Gamma_0$, with $J \neq \emptyset$, it is $inf\, \mathcal{G}_J|\mathcal{H}_J \leq 0$. Given any $J \subseteq \Gamma_0$, as $J \notin \mathcal{K}$, it is $inf\, \mathcal{G}_n|\mathcal{H}_J \leq 0$, for every $s_1, \ldots, s_n$. Moreover, $\mathcal{G}_n|\mathcal{H}_J = \mathcal{G}_J|\mathcal{H}_J + \mathcal{G}_{J_n \setminus J}|\mathcal{H}_J$; in particular, if we choose $s_i = 0$ for $i \notin J$, it is $\mathcal{G}_n|\mathcal{H}_J = \mathcal{G}_J|\mathcal{H}_J$. Then, in order the condition $inf\, \mathcal{G}_n|\mathcal{H}_J \leq 0\,,\ \ \forall\, s_1, \ldots, s_n$, be satisfied, it must be $inf\, \mathcal{G}_J|\mathcal{H}_J \leq 0$, for every s_i, $i \in J$. Therefore, the assessment $\mathcal{M}_{\Gamma_0}$ on $\mathcal{F}_{\Gamma_0}$ is coherent. □

Remark 1. We observe that $inf\, \mathcal{G}_J|\mathcal{H}_J > 0$ for some s_i, with $i \in J$, implies $inf\, \mathcal{G}_n|\mathcal{H}_J > 0$ with the same s_i, for $i \in J$, and $s_i = 0$, for $i \in J_n \setminus J$.

We give below a necessary and sufficient condition of coherence.

Theorem 6. Let be given a family $\mathcal{F}_n = \{X_i|Y_i, i \in J_n\}$ of n conditional random quantities and a conditional prevision assessment $\mathcal{M}_n = (\mu_i, i \in J_n)$ on $\mathcal{F}_n$. Moreover, let K^* be any non empty subset of J_n such that $K_0 \subseteq K^*$. The assessment $\mathcal{M}_n$ is coherent if and only if:
(i) $inf\, \mathcal{G}_n|\mathcal{H}_n \cdot sup\, \mathcal{G}_n|\mathcal{H}_n \leq 0\ \forall\ s_i \in \mathbb{R}, i \in J_n$; (ii) $\mathcal{M}_{K^*}$ on $\mathcal{F}_{K^*}$ is coherent.

Proof. Of course, coherence of $\mathcal{M}_n$ implies (i) and (ii). Conversely, based on Definition 4, we have to prove that, for every $K \subseteq J_n$, it is $inf\, \mathcal{G}_K|\mathcal{H}_K \leq 0$. We distinguish two cases: (a) $K \subseteq K^*$; (b) $K \nsubseteq K^*$. In the case (a) the condition $inf\, \mathcal{G}_K|\mathcal{H}_K \leq 0$ follows from coherence of $\mathcal{M}_{K^*}$; in the case (b), $K \nsubseteq K_0$ and hence $K \notin \mathcal{K}$; therefore $inf\, \mathcal{G}_n|\mathcal{H}_K \leq 0$. Then, by reasoning as in Theorem 5, it follows $inf\, \mathcal{G}_K|\mathcal{H}_K \leq 0$. Therefore $\mathcal{M}_n$ is coherent. □

We illustrate the previous result by the following

Example 2. Given a random vector (X_1, X_2, Y_1, Y_2), assume that the constituents are

$$\begin{array}{ll} C_1 = (X_1 = 1, X_2 = 0, Y_1 = 0, Y_2 = 1), & C_2 = (X_1 = 1, X_2 = 0, Y_1 = 1, Y_2 = 1), \\ C_3 = (X_1 = 0, X_2 = 0, Y_1 = 1, Y_2 = 1), & C_4 = (X_1 = 1, X_2 = 2, Y_1 = 0, Y_2 = 0), \\ C_5 = (X_1 = 1, X_2 = 2, Y_1 = 1, Y_2 = 0), & C_6 = (X_1 = 0, X_2 = 2, Y_1 = 1, Y_2 = 0). \end{array}$$

Then, consider the assessment $\mathcal{M}_3 = (0, 1, 0)$ on $\mathcal{F}_3 = \{X_1|Y_1, X_2|Y_2, Y_2|X_2\}$. We observe that $\mathcal{H}_3 = (Y_1 \neq 0) \vee (Y_2 \neq 0) \vee (X_2 \neq 0) = \Omega$ and $\mathcal{G}_3|\mathcal{H}_3 = \mathcal{G}_3 = s_1 Y_1 X_1 + s_2 Y_2 (X_2 - 1) + s_3 X_2 Y_2$. The values of $\mathcal{G}_3|\mathcal{H}_3$ are

$$g_1 = -s_2\,,\ g_2 = s_1 - s_2\,,\ g_3 = -s_2\,,\ g_4 = 0\,,\ g_5 = s_1\,,\ g_6 = 0\,.$$

Now, it can be verified that $inf\, \mathcal{G}_3|H_1 \leq 0$ and $inf\, \mathcal{G}_3|H_3 \leq 0$ for all s_1, s_2, s_3, which means that $\{1, 3\} \subseteq \Gamma_0$. On the contrary, for some s_1, s_2, s_3 (e.g. for $s_2 > 0$, $s_1 < s_2$) it is $inf\, \mathcal{G}_3|H_2 \cdot sup\, \mathcal{G}_3|H_2 = -s_2(s_1 - s_2) > 0$. Thus, $\Gamma_0 = \{1, 3\}$ and $K_0 = \{2\}$. Moreover, $\mathcal{G}_{K_0}|\mathcal{H}_{K_0} = s_2 Y_2 (X_2 - \mu_2)|H_2 = -s_2$; hence the condition $inf\, \mathcal{G}_{K_0}|\mathcal{H}_{K_0} \leq 0$ is not satisfied for every s_2. This means that condition (ii) is not satisfied, i.e. the assessment $\mu_2 = 1$ on $X_2|Y_2$ is not coherent, so that $\mathcal{M}_3$ is not coherent too.
Of course, by Theorem 5, the assessment $(0, 0)$ on $\{X_1|Y_1, Y_2|X_2\}$ is coherent.

6 A procedure for checking coherence

In this section, based on a suitable alternative theorem, we characterize the coherence of conditional prevision assessments by some theoretical results; then we propose an algorithm for the checking of coherence.
Let $\mathbf{z}$, $\mathbf{s}$ and A be, respectively, a row $m-$vector, a column $n-$vector and a $m \times n-$matrix. The vector $\mathbf{z} = (z_1, \ldots, z_m)$ is said *semipositive* if $z_i \geq 0, \forall\, i \in J_m$ and $z_1 + \cdots + z_m > 0$. Then, we have (Gale 1960; Theorem 2.9)

Theorem 7. Exactly one of the following alternatives holds.
(i) the equality $\mathbf{z}A = 0$ has a *semipositive* solution;
(ii) the inequality $A\mathbf{s} > 0$ has a solution.

We observe that the equality $\mathbf{z}A = 0$ has a *semipositive* solution $\mathbf{z} = (z_1, \ldots, z_m)$ if and only if the equality $\mathbf{p}A = 0$ has a *semipositive* solution $\mathbf{p} = (p_1, \ldots, p_m)$ with $p_1 + \cdots + p_m = 1$.
Given two random vectors $X = (X_1, \ldots, X_n), Y = (Y_1, \ldots, Y_n)$, we set $(X, Y) = (X_1, \ldots, X_n, Y_1, \ldots, Y_n)$; moreover, we denote by $\mathcal{C}_{XY}$ the realm of (X, Y), that is the (finite) set of points $(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^{2n}$ such that $(X = \mathbf{x}, Y = \mathbf{y}) \neq \emptyset$. We recall that $H_i = (Y_i \neq 0),\ i \in J_n,\ \mathcal{H}_n = H_1 \vee \cdots \vee H_n$; moreover, we denote by $\mathcal{C}^0_{XY} = \{(\mathbf{x}_1, \mathbf{y}_1), \ldots, (\mathbf{x}_m, \mathbf{y}_m)\}$, with $(\mathbf{x}_r, \mathbf{y}_r) = (x_{r1}, \ldots, x_{rn}, y_{r1}, \ldots, y_{rn}),\ r \in J_m$, the subset of points $(\mathbf{x}, \mathbf{y})$ of $\mathcal{C}_{XY}$ such that $\mathbf{y} \neq \mathbf{0}$; this means that, for any $(\mathbf{x}, \mathbf{0}) \in \mathcal{C}_{XY}$, it is $(\mathbf{x}, \mathbf{0}) \notin \mathcal{C}^0_{XY}$. Given an assessment $\mathcal{M}_n = (\mu_1, \ldots, \mu_n)$ on $F_n = \{X_1|Y_1, \ldots, X_n|Y_n\}$, we denote by C_r the constituent $(X = \mathbf{x}_r, Y = \mathbf{y}_r)$. Then, the value g_r of the random gain $\mathcal{G}_n|\mathcal{H}_n = \sum_{i=1}^n s_i Y_i(X_i - \mu_i)$, associated with the constituent C_r, is given by

$$g_r = \sum_{i=1}^n s_i y_{ri}(x_{ri} - \mu_i) = \sum_{i=1}^n s_i(x_{ri}y_{ri} - \mu_i y_{ri})\,,\ r \in J_m\,.$$

We define the matrix $A = (a_{ri})$, where $a_{ri} = x_{ri}y_{ri} - \mu_i y_{ri}$, $r \in J_m$, $i \in J_n$, and the column $n-$vector $\mathbf{s} = (s_1, \ldots, s_n)^t$. If the inequality $A\mathbf{s} > 0$ has a solution, this means $g_r > 0,\ \forall\, r$; that is $inf\, \mathcal{G}_n|\mathcal{H}_n > 0$. Then, by (the alternative) Theorem 7, the coherence condition $inf\, \mathcal{G}_n|\mathcal{H}_n \leq 0,\ \forall\, s_1, \ldots, s_n$, means that the equality $\mathbf{z}A = 0$ has a *semipositive* solution $\mathbf{p} = (p_1, \ldots, p_m)$, with $\sum_{r=1}^m p_r = 1$. This amounts to solvability of the following system

$$\left\{ \begin{array}{l} \sum_{r=1}^m p_r(x_{ri}y_{ri} - \mu_i y_{ri}) = 0\,,\ i \in J_n, \\ \sum_{r=1}^m p_r = 1\,;\ \ p_r \geq 0,\ r \in J_m. \end{array} \right. \tag{5}$$

Remark 2. Given any $K \subset J_n$, we denote by $A_K = (a_{ri})$ the sub-matrix of A such that $i \in J_n$ and r such that $C_r \subseteq \mathcal{H}_K$. By the same alternative theorem, we have that the condition $inf\, \mathcal{G}_n|\mathcal{H}_K \leq 0,\ \forall\, s_1, \ldots, s_n$, means that the inequality $A_K\mathbf{s} > 0$ has no solutions, or equivalently that the equality $\mathbf{p}_K A_K = 0$ has a semipositive solution $\mathbf{p}_K = (p_r, r : C_r \subseteq \mathcal{H}_K)$; i.e., the following system is solvable

$$\left\{ \begin{array}{l} \sum_{r: C_r \subseteq \mathcal{H}_K} p_r(x_{ri}y_{ri} - \mu_i y_{ri}) = 0\,,\ i \in J_n, \\ \sum_{r: C_r \subseteq \mathcal{H}_K} p_r = 1\,;\ \ p_r \geq 0,\ r : C_r \subseteq \mathcal{H}_K. \end{array} \right. \tag{6}$$

We observe that, denoting by $(x_j^{(i)}, y_j^{(i)})$ the generic possible value of (X_i, Y_i), the system (5) can be equivalently rewritten as

$$\left\{ \begin{array}{l} \sum_j x_j^{(i)} y_j^{(i)} \sum_{r:C_r \subseteq (X_i = x_j^{(i)}, Y_i = y_j^{(i)})} p_r = \mu_i \sum_j y_j^{(i)} \sum_{r:C_r \subseteq (Y_i = y_j^{(i)})} p_r, \;\; i \in J_n, \\ \sum_{r=1}^m p_r = 1\,; \;\; p_r \geq 0, \; r \in J_m. \end{array} \right. \tag{7}$$

Notice that, in probabilistic terms, we have the following interpretations

$$\begin{array}{l} p_r = P(C_r|\mathcal{H}_n) = P[(X = \mathbf{x}_r, Y = \mathbf{y}_r)|\mathcal{H}_n]\,; \\ \sum_{r:C_r \subseteq (X_i = x_j^{(i)}, Y_i = y_j^{(i)})} p_r = P[(X_i = x_j^{(i)}, Y_i = y_j^{(i)})|\mathcal{H}_n]\,; \\ \sum_{r:C_r \subseteq (Y_i = y_j^{(i)})} p_r = P[(Y_i = y_j^{(i)})|\mathcal{H}_n]\,; \end{array} \tag{8}$$

hence, system (7) can be looked at

$$\Pr(X_i Y_i|\mathcal{H}_n) = \mu_i Pr(Y_i|\mathcal{H}_n), \; i \in J_n\,; \;\; P(\mathcal{H}_n|\mathcal{H}_n) = 1\,. \tag{9}$$

Now, assuming that system (7) is solvable, we denote by S its (non empty) set of solutions. Given any $\mathbf{p} = (p_1, \ldots, p_m) \in S$, we set

$$\Phi_j(\mathbf{p}) = \sum_{r:C_r \subseteq H_j} p_r, \; M_j = max_{\mathbf{p} \in S}\, \Phi_j(\mathbf{p}), \; j \in J_n; \;\; I_0 = \{j \in J_n : M_j = 0\}. \tag{10}$$

Of course, solvability of system (7) implies $I_0 \subset J_n$. Given any $K \subseteq J_n$, we denote by $(\mathcal{F}_K, \mathcal{M}_K)$ the pair associated with K and by $\mathcal{G}_K|\mathcal{H}_K$ (resp., by $(\mathcal{S}_K)$) the random gain (resp., the system) associated with $(\mathcal{F}_K, \mathcal{M}_K)$.
Of course, $\mathcal{G}_n = \mathcal{G}_{J_n}$ and $\mathcal{F}_n = \mathcal{F}_{J_n}$. We have

Theorem 8. Assume that system (7) is solvable; moreover, let I_0 be defined as in (10). Then, given any $K \subset J_n$ such that $K \setminus I_0 \neq \emptyset$, the system $(\mathcal{S}_K)$ is solvable; that is $inf\, \mathcal{G}_K|\mathcal{H}_K \leq 0$. Moreover, the sub-assessment $\mathcal{M}_{J_n \setminus I_0}$ on the sub-family $\mathcal{F}_{J_n \setminus I_0}$ is coherent.

Proof. Given any $j \in K \setminus I_0$ there exists a solution $\mathbf{p}^{(j)} = (p_1^{(j)}, \ldots, p_m^{(j)}) \in S$ such that $\Phi_j(\mathbf{p}^{(j)}) > 0$; moreover

$$\sum_{r:C_r \subseteq \mathcal{H}_K} p_r^{(j)} \geq \sum_{r:C_r \subseteq H_j} p_r^{(j)} = \Phi_j(\mathbf{p}^{(j)}) > 0\,.$$

Hence, $\mathbf{p}^{(j)}$ is a solution of the following system related with system (7)

$$\left\{ \begin{array}{l} \sum_j x_j^{(i)} y_j^{(i)} \sum_{r:C_r \subseteq (X_i = x_j^{(i)}, Y_i = y_j^{(i)})} p_r = \mu_i \sum_j y_j^{(i)} \sum_{r:C_r \subseteq (Y_i = y_j^{(i)})} p_r, \;\; i \in K, \\ \sum_{r:C_r \subseteq \mathcal{H}_K} p_r > 0\,; \;\; p_r \geq 0, \; r \in J_m. \end{array} \right. \tag{11}$$

As it can be verified, the solvability of the system (11) is equivalent to solvability of the system $(\mathcal{S}_K)$; that is, by the alternative theorem, to satisfiability of the condition $inf\, \mathcal{G}_K|\mathcal{H}_K \leq 0$. In particular, the condition $inf\, \mathcal{G}_K|\mathcal{H}_K \leq 0$ holds for every $K \subseteq J_n \setminus I_0$ and this amounts to coherence of $\mathcal{M}_{J_n \setminus I_0}$. □

By the previous result, we obtain

Theorem 9. Let be given a family $\mathcal{F}_n = \{X_i|Y_i, i \in J_n\}$ of n conditional random quantities and a conditional prevision assessment $\mathcal{M}_n = (\mu_i, i \in J_n)$ on $\mathcal{F}_n$. Moreover, let K^* be any non empty subset of J_n such that $I_0 \subseteq K^*$. The assessment $\mathcal{M}_n$ is coherent if and only if:
(i) the system (7) is solvable; (ii) $\mathcal{M}_{K^*}$ on $\mathcal{F}_{K^*}$ is coherent.

Proof. Of course, coherence of $\mathcal{M}_n$ implies conditions (i) and (ii). Conversely, based on Definition 4, we have to prove that $inf\, \mathcal{G}_K|\mathcal{H}_K \leq 0, \forall\, K \subseteq J_n$. We observe that, by (i), it is $inf\, \mathcal{G}_n|\mathcal{H}_n \leq 0$ and $I_0 \subset J_n$. We distinguish two cases: (a) $K \subseteq K^*$; (b) $K \nsubseteq K^*$. In the case (a) the condition $inf\, \mathcal{G}_K|\mathcal{H}_K \leq 0$ follows from coherence of $\mathcal{M}_{K^*}$; in the case (b) the condition $inf\, \mathcal{G}_K|\mathcal{H}_K \leq 0$ follows by Theorem 8, as $K \setminus K^* \neq \emptyset$. □

Remark 3. We recall that, for each $r \in J_m$, C_r represents the constituent $(X = \mathbf{x}_r, Y = \mathbf{y}_r)$; hence, given any $K \subseteq J_n$, with $K \setminus I_0 \neq \emptyset$, for each r such that $C_r \subseteq (Y_i = y_j^{(i)})\,,\ i \in K$, we have $C_r \subseteq \mathcal{H}_K$. Hence, in system (11) for all the variables p_r's it is $C_r \subseteq \mathcal{H}_K$ and the condition $r \in J_m$ can be replaced by $r : C_r \subseteq \mathcal{H}_K$. It follows that, by defining

$$\lambda_r = \frac{p_r}{\sum_{r:C_r\subseteq\mathcal{H}_K} p_r}\,,\ \forall\, r : C_r \subseteq \mathcal{H}_K\,,$$

the system (11) can be rewritten as the following one

$$\begin{cases} \sum_j x_j^{(i)} y_j^{(i)} \sum_{r:C_r\subseteq(X_i=x_j^{(i)},Y_i=y_j^{(i)})} \lambda_r = \mu_i \sum_j y_j^{(i)} \sum_{r:C_r\subseteq(Y_i=y_j^{(i)})} \lambda_r, \quad i \in K, \\ \sum_{r:C_r\subseteq\mathcal{H}_K} \lambda_r = 1\,;\quad \lambda_r \geq 0,\ r : C_r \subseteq \mathcal{H}_K. \end{cases} \tag{12}$$

Moreover, concerning system (6), as $s_1, \ldots, s_n$ are arbitrary, by choosing $s_i = 0, \forall\, i \in J_n \setminus K$, the system obtained by (6), with $i \in J_n$ replaced by $i \in K$, is equivalent to system (11). As a consequence: (i) $K_0 = I_0$; (ii) Theorems 5 and 8 are equivalent; (iii) Theorems 6 and 9 are equivalent too.

We observe that, if $K \subseteq I_0$, nothing can be said about the solvability of system $(\mathcal{S}_K)$, which requires a direct checking, by starting with $K = I_0$.
Based on Theorems 8 and 9, we can use the algorithm below for the checking of coherence.

Algorithm 1. Let be given a conditional prevision assessment $\mathcal{M}_n = (\mu_1, \ldots, \mu_n)$ on $\mathcal{F}_n = \{X_1|Y_1, \ldots, X_n|Y_n\}$.
Step 1. Check the solvability of system (7); if the system is not solvable, then $\mathcal{M}_n$ is not coherent.
Step 2. If the system is solvable, determine I_0; if $I_0 = \emptyset$, then $\mathcal{M}_n$ is coherent.
Step 3. If $I_0 \neq \emptyset$, then determine the pair $(\mathcal{F}_{I_0}, \mathcal{M}_{I_0})$; replace the pair $(\mathcal{F}_n, \mathcal{M}_n)$ by $(\mathcal{F}_{I_0}, \mathcal{M}_{I_0})$ and repeat the previous steps.

As we can see, using the algorithm above, we can check coherence of the assessment $\mathcal{M}_n$ on $\mathcal{F}_n$ in a finite number of iterations. If the initial system is solvable, a suitable sequence of sets $I_0^{(1)}, \ldots, I_0^{(t)}$ is computed. We have two cases: (a) if $\mathcal{M}_n$ is coherent, it is $t \leq n$ and $I_0^{(t)} = \emptyset$; (b) if $\mathcal{M}_n$ is not coherent, it is $t \leq n-1$ and $I_0^{(t)} \neq \emptyset$. We give an example to illustrate Algorithm 1.

Example 3. *(we continue Example 2)* Concerning the assessment $\mathcal{M}_3 = (0,1,0)$ on the family $\mathcal{F}_3 = \{X_1|Y_1, X_2|Y_2, Y_2|X_2\}$, with each constituent C_r, we associate a variable p_r, $r = 1\ldots,6$. Then, based on Algorithm 1, we check the solvability of the initial system given below.

$$\begin{cases} 0(p_1+p_3+p_4+p_6)+1(p_2+p_5) = 0(0(p_1+p_4)+1(p_2+p_3+p_5+p_6)), \\ 0 = 1(0(p_4+p_5+p_6)+1(p_1+p_2+p_3)), \\ 0 = 0(0(p_1+p_2+p_3)+2(p_4+p_5+p_6)), \\ \sum_{r=1}^{6} p_r = 1, \;\; p_r \geq 0, \; r = 1,\ldots,6, \end{cases} \tag{13}$$

which can be written

$$\begin{cases} p_2+p_5 = 0\,, \;\; p_1+p_2+p_3 = 0\,, \;\; 0 = 0\,, \\ \sum_{r=1}^{6} p_r = 1, \;\; p_r \geq 0, \; r = 1,\ldots,6. \end{cases} \tag{14}$$

Each vector $\mathbf{p} = (p_1,\ldots,p_6)$, with $p_1 = p_2 = p_3 = p_5 = 0, p_4+p_6 = 1$, is a solution of this system. We have

$$\Phi_1(\mathbf{p}) = p_2+p_3+p_5+p_6\,,\; \Phi_2(\mathbf{p}) = p_1+p_2+p_3\,,\; \Phi_3(\mathbf{p}) = p_4+p_5+p_6\,,$$

hence $M_1 > 0$, $M_2 = 0$, $M_3 > 0$. Then, $I_0 = \{2\}$ and we have to check the coherence of the assessment $\mu_2 = \mathbb{P}(X_2|Y_2) = 1$. As conditionally on $(Y_2 \neq 0)$ the unique possible value of X_2 is 0, it must be $\mathbb{P}(X_2|Y_2) = 0$; hence, by the algorithm it results that the assessment $\mathcal{M}_3$ is not coherent. Of course, by Theorem 8, the sub-assessment $(0,0)$ on $\{X_1|Y_1, Y_2|X_2\}$ is coherent.

7 Imprecise conditional prevision assessments

In this section we briefly examine imprecise conditional prevision assessments; we introduce below the notions of generalized coherence and total coherence.

Definition 5. Let be given any random quantities $X_1,\ldots,X_n,Y_1,\ldots,Y_n$ and a set $\mathcal{S} \subseteq \mathbf{R}^n$. With each point $\mathcal{M}_n = (\mu_1,\ldots,\mu_n) \in \mathcal{S}$ we associate the family $\mathcal{F}_n = \{X_1|Y_1,\ldots,X_n|Y_n\}$, where $X_i|Y_i = \mu_i + Y_i(X_i - \mu_i)$, $i \in J_n$. We say that the set $\mathcal{S}$ is coherent in a generalized sense (*g-coherent*) if and only if there exists $\mathcal{M}_n \in \mathcal{S}$ which is a coherent conditional prevision assessment on $\mathcal{F}_n$.
We say that the set $\mathcal{S}$ is *totally coherent* if and only if, for every $\mathcal{M}_n \in \mathcal{S}$, $\mathcal{M}_n$ is a coherent conditional prevision assessment on $\mathcal{F}_n$.

Of course, total coherence implies g-coherence.

Given a family of n conditional random quantities $\mathcal{F}_n = \{X_1|Y_1, \ldots, X_n|Y_n\}$, we assume $(X_i, Y_i) \in \mathcal{C}_i\,,\ \ i \in J_n$; moreover, for each i, we set $\mathcal{C}_i^0 = (x, y) \in \mathcal{C}_i : y \neq 0$ and $H_i = (Y_i \neq 0)$. For each i, we denote by X_i^0 the set of values of X_i such that for each $x \in X_i^0$ there exists a possible value y of Y_i such that $(x, y) \in \mathcal{C}_i^0$. Moreover, we set $m_i = min\, X_i^0\,,\ \ M_i = max\, X_i^0\,,\ \ i \in J_n$. We recall that, assuming $Y_i \geq 0$, or $Y_i \leq 0$, the assessment $\mathbb{P}(X_i|Y_i) = \mu_i$ is coherent if and only if $m_i \leq \mu_i \leq M_i$. We set $I = [m_1, M_1] \times \cdots \times [m_n, M_n]$. Then, we have the following result which concerns the total coherence of I.

Theorem 10. Let be given a conditional prevision assessment $\mathcal{M}_n = (\mu_1, \ldots, \mu_n)$ on a family $\mathcal{F}_n = \{X_1|Y_1, \ldots, X_n|Y_n\}$, where for each i it is $Y_i \geq 0$, or $Y_i \leq 0$. Moreover, assume that $H_i H_j = \emptyset$, for each $i \neq j$. Then, the assessment $\mathcal{M}_n$ is coherent if and only if $m_i \leq \mu_i \leq M_i$ for every i; that is, I is totally coherent.

Proof. We set $G_i = s_i Y_i (X_i - \mu_i),\ i \in J_n$; then

$$\mathcal{G}_n = G_1 + \cdots + G_n = H_1 G_1 + \cdots + H_n G_n\,,$$

where $s_1, \ldots, s_n$ are arbitrary real numbers. Of course, for each i, the condition $inf\ G_i|H_i \leq 0\, \forall\, s_i$ is satisfied if and only if $m_i \leq \mu_i \leq M_i$. Then, recalling that $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$, from the hypothesis $H_i H_j = \emptyset$ for $i \neq j$, it follows

$$\mathcal{G}_n|\mathcal{H}_n = \left\{ \begin{array}{ll} G_1|H_1, & H_1\ true, \\ \ldots\ldots & \ldots\ldots \\ G_n|H_n, & H_n\ true. \end{array} \right.$$

Then

$$inf\ \mathcal{G}_n|\mathcal{H}_n = min\ \{inf\ G_i|H_i\,,\ i \in J_n\}\,,$$

and the condition $inf\ \mathcal{G}_n|\mathcal{H}_n \leq 0\ \forall\, s_1, \ldots, s_n$, is satisfied if and only if it is satisfied the condition $inf\ G_i|H_i \leq 0\, \forall\, s_i,\ i \in J_n$; that is $m_i \leq \mu_i \leq M_i\, \forall\, s_i,\ i \in J_n$. Of course, a similar reasoning can be applied to each sub-family of $\mathcal{F}_n$; hence I is totally coherent. □

We illustrate the previous result by the following

Example 4. Assume that the random vector $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ has the following possible values

$$(1,1,1,1,0,0)\,,\quad (-1,-1,-1,1,0,0)\,,\quad (1,1,1,0,1,0)\,,$$

$$(-1,-1,-1,0,1,0)\,,\quad (1,1,1,0,0,1)\,,\quad (-1,-1,-1,0,0,1)\,;$$

moreover, let $\mathcal{M} = (\mu_1, \mu_2, \mu_3)$ a conditional prevision assessment on $\mathcal{F}_3 = \{X_1|Y_1, X_2|Y_2, X_3|Y_3\}$. We observe that $[m_i, M_i] = [-1, 1]$, $i = 1, 2, 3$, and $I = [-1, 1]^3$. Moreover, we have the following values for the random gain $\mathcal{G}_3|\mathcal{H}_3$

$$s_1(1-\mu_1)\,,\ -s_1\mu_1\,,\ s_2(1-\mu_2)\,,\ -s_2\mu_2\,,\ \ s_3(1-\mu_3)\,,\ -s_3\mu_3\,.$$

As it can be easily verified, the condition $min\,\mathcal{G}_3|\mathcal{H}_3 \leq 0\,,\ \forall\, s_1, s_2, s_3$, is satisfied if and only if $-1 \leq \mu_i \leq 1$, $i = 1, 2, 3$; of course, a similar reasoning can be applied to each subfamily of $\mathcal{F}_3$. Hence the interval $I = [-1, 1]^3$ is totally coherent.

8 Conclusions

In this paper we have introduced the notion of coherence for conditional prevision assessments on finite families of general conditional random quantities. Moreover, we have examined the compound prevision theorem and the relation between $\mathbb{P}(X|Y)$ and $\mathbb{P}(Y|X)$. Then, we have given some theoretical results on random gains and, based on a suitable alternative theorem, we have given a characterization of coherence. We have also proposed an algorithm for the checking of coherence. Finally, we have introduced the notions of generalized and total coherence; then, we have briefly examined the case of imprecise conditional prevision assessments. To illustrate our results we have considered some examples. Future work should concern the deepening of the case of imprecise prevision assessments.

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