Quasi Conjunction and p-Entailment in Nonmonotonic Reasoning

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Abstract

We study, in the setting of coherence, the extension of a probability assessment defined on n conditional events to their quasi conjunction. We consider, in particular, two special cases of logical dependencies; moreover, we examine the relationship between the notion of p-entailment of Adams and the inclusion relation of Goodman and Nguyen. We also study the probabilistic semantics of the QAND rule of Dubois and Prade; then, we give a theoretical result on p-entailment. **Keywords.** Coherence, Lower/upper probability bounds, Quasi conjunction, QAND rule, p-entailment.

1 Introduction

In classical (monotonic) logic, if a conclusion C follows from some premises, then C also follows when the set of premises is enlarged; that is, adding premises never invalidates any conclusions. Differently, in (nonmonotonic) commonsense reasoning typically we are in a situation of partial knowledge and a conclusion reached from a set of premises may be retracted, when some premises are added. Nonmonotonic reasoning is a relevant topic in the field of artificial intelligence and has been studied in literature by many, symbolic or numerical, formalisms (see, e.g. [2, 3, 4, 9]). A remarkable theory, related with nonmonotonic reasoning, has been given by Adams in his probabilistic logic of conditionals ([1]). We recall that the approach of Adams can be developed with full generality by exploiting a coherence-based probabilistic reasoning, which allows a direct assignment of conditional probabilities, without assuming that conditioning events have a positive probability ([5]). A basic notion in the work of Adams is the quasi conjunction of conditionals, which also plays a relevant role in the work of Dubois and Prade on conditional objects, where a suitable

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QAND rule is introduced to characterize entailment from a knowledge base. In our paper we deepen some probabilistic aspects related with QAND rule and with the conditional probabilistic logic of Adams.
The paper is organized as follows: In Section 2 we recall the p-consistency and p-entailment notions in the setting of coherence; in Sections 3 and 4 we study the lower and upper probability bounds for the quasi conjunction of conditional events, by relating them to Lukasiewicz t-norm and Hamacher t-conorm, respectively; we also examine two special cases of logical dependencies related with the inclusion relation of Goodman and Nguyen and with the compound probability theorem; in Section 5 we deepen the analysis on the lower and upper probability bounds for the quasi conjunction, by examining further aspects; in Section 6 we examine the relation between the notion of p-entailment and the inclusion relation of Goodman and Nguyen; then, we study the probabilistic semantics of *QAND* rule, by proving the p-entailment from any given family of conditional events $\mathcal{F}$ to the quasi conjunction $\mathcal{C}(\mathcal{F})$; finally, we prove the equivalence between p-entailment from $\mathcal{F}$ and p-entailment from the $\mathcal{C}(\mathcal{S})$, for some non-empty subset $\mathcal{S}$ of $\mathcal{F}$; in Section 7 we give some conclusions. Due to the lack of space (almost) all proofs of our results are omitted.

2 Some Preliminary Notions

In this section we recall, in the setting of coherence ([5, 6]), the notions of p-consistency and p-entailment of Adams ([1]). Given a conditional knowledge base $\mathcal{K}_n = \{H_i|\!\sim E_i\,,\ i = 1,\dots,n\}$, we denote by $\mathcal{F}_n = \{E_i|H_i\,,\ i = 1,\dots,n\}$ the associated family of conditional events.

Definition 1. *The knowledge base $\mathcal{K}_n = \{H_i|\!\sim E_i\,,\ i = 1,\dots,n\}$ is* p-consistent *iff, for every set of lower bounds $\{\alpha_i, i = 1,\dots,n\}$, with $\alpha_i \in [0,1)$, there exists a coherent probability assessment $\{p_i, i = 1,\dots,n\}$ on $\mathcal{F}_n$, with $p_i = P(E_i|H_i)$, such that $p_i \geq \alpha_i, i = 1,\dots,n$.*

We say that $\mathcal{F}_n$ is p-consistent when it is p-consistent the associated knowledge base $\mathcal{K}_n$; then, we point out that the property of p-consistency for $\mathcal{F}_n$ is equivalent to the coherence of the assessment $(p_1, p_2, \dots, p_n) = (1, 1, \dots, 1)$ on $\mathcal{F}_n$ (strict p-consistency, [5]).

Definition 2. *A p-consistent knowledge base $\mathcal{K}_n = \{H_i|\!\sim E_i\,,\ i = 1,\dots,n\}$* p-entails *the conditional $A|\!\sim B$, denoted $\mathcal{K}_n \ \Rightarrow_p \ A|\!\sim B$, iff there exists a non-empty subset $\Gamma \subseteq \{1,\dots,n\}$ such that, for every $\alpha \in [0,1)$, there exists a set of lower bounds $\{\alpha_i, i \in \Gamma\}$, with $\alpha_i \in [0,1)$, such that for all coherent probability assessments $\{z, p_i, i \in \Gamma\}$ defined on $\{B|A, E_i|H_i\,,\ i \in \Gamma\}$, with $z = P(B|A)$ and $p_i = P(E_i|H_i)$, if $p_i \geq \alpha_i$ for every $i \in \Gamma$, then $z \geq \alpha$.*

Remark 1. *We say that a family of conditional events $\mathcal{F}_n$ p-entails a conditional event $B|A$ when the associated knowledge base $\mathcal{K}_n$ p-entails the conditional $A|\!\sim B$. Therefore, p-entailment of $B|A$ from $\mathcal{F}_n$ amounts to the existence of a non-empty subset $\mathcal{S} = \{E_i|H_i,\ i \in \Gamma\}$ of $\mathcal{F}_n$ such that, defining $P(E_i|H_i) = p_i, P(B|A) = z$, for every $\alpha \in [0,1)$, there exist lower bounds $\alpha_i, i \in \Gamma$, with $\alpha_i \in [0,1)$, such that $p_i \geq \alpha_i,\ i \in \Gamma$, implies $z \geq \alpha$.*

3 Lower and Upper Bounds for Quasi Conjunction

Let A, H, B, K be logically independent events, with $H \neq \emptyset, K \neq \emptyset$. The quasi conjunction of two conditional events $A|H$ and $B|K$, as defined in ([1]), is given by $\mathcal{C}(A|H, B|K) = (AH \vee H^c) \wedge (BK \vee K^c)|(H \vee K)$. We recall that quasi conjunction plays a key role in the logic of conditional objects ([4]).
It can be easily verified that, for every pair (x, y), with $x \in [0,1], y \in [0,1]$, the probability assessment (x, y) on $\{A|H, B|K\}$ is coherent. Then, it can be verified (see [6]) that, for each given assessment (x, y) on $\{A|H, B|K\}$, the probability assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F} = \{A|H, B|K, \mathcal{C}(A|H, B|K)\}$, with $z = P[\mathcal{C}(A|H, B|K)]$, is a coherent extension of (x, y) if and only if

$$\max(x + y - 1, 0) = l \leq z \leq u = \begin{cases} \frac{x+y-2xy}{1-xy}, & (x, y) \neq (1, 1), \\ 1, & (x, y) = (1, 1). \end{cases}$$

We observe that the lower bound l coincides with the Lukasiewicz t-norm $T_L(x, y)$, while the upper bound u coincides with the Hamacher t-conorm $S_0^H(x, y)$, with parameter $\lambda = 0$ (see [8]).

Remark 2. *Notice that, if the events A, B, H, K were not logically independent, then some constituents C_h's (at least one) would become impossible and the lower bound l could increase, while the upper bound u could decrease. To examine this aspect we will consider two special cases of logical dependencies.*

3.1 The Case $A|H \subseteq B|K$

We recall the Goodman & Nguyen relation of inclusion for conditional events ([7]). Given two conditional events $A|H$ and $B|K$, we say that $A|H$ implies $B|K$, denoted by $A|H \subseteq B|K$, if and only if $AH \subseteq BK$ and $B^cK \subseteq A^cH$. Given any conditional events $A|H, B|K$, we denote by Π_x the set of coherent probability assessment x on $A|H$, by Π_y the set of coherent probability assessment y on $B|K$ and by Π the set of coherent probability assessment (x, y) on $\{A|H, B|K\}$; moreover we indicate by $T_{x \leq y}$ the triangle $\{(x, y) \in [0,1]^2 : x \leq y\}$. In the next result, to avoid the specific analysis of some trivial cases, we assume $\Pi_x = \Pi_y = [0,1]$. We have

Theorem 1. *Let $A|H, B|K$ be two conditional events, with $\Pi_x = \Pi_y = [0,1]$. Then: $A|H \subseteq B|K \Longleftrightarrow \Pi \subseteq T_{x \leq y}$.*

Actually, concerning Theorem 1, the implication $\Longrightarrow$ also holds in trivial cases where $\Pi_x \subset [0,1]$, or $\Pi_y \subset [0,1]$.

Remark 3. *We observe that, under the hypothesis $A|H \subseteq B|K$, we have $\mathcal{C}(A|H, B|K) = (AH \vee H^cBK)\,|\,(H \vee K)$ and, as we can verify, it is*

$$A|H \subseteq \mathcal{C}(A|H, B|K) \subseteq B|K\,. \tag{1}$$

Moreover, if we do not assume further logical relations, then $\Pi = T_{x \leq y}$ and, for each coherent assessment (x, y) on $\{A|H, B|K\}$, the extension $z = P[\mathcal{C}(A|H, B|K)]$ is coherent if and only if $l \leq z \leq u$, where

$$l = x = \min(x, y)\,,\ u = y = \max(x, y)\,.$$

We remark that the values l, u may change if we add further logical relations; in particular, if $H = K$, it is $\mathcal{C}(A|H, B|H) = A|H$, in which case $l = u = x$.

Finally, in agreement with Remark 2, we observe that

$$T_L(x, y) \leq \min(x, y) \leq \max(x, y) \leq S_0^H(x, y).$$

3.2 Compound Probability Theorem

We now examine the quasi conjunction of $A|H$ and $B|AH$, with A, B, H logically independent events. As it can be easily verified, we have $\mathcal{C}(A|H, B|AH) = AB|H$; moreover, the probability assessment (x, y) on $\{A|H, B|AH\}$ is coherent, for every $(x, y) \in [0, 1]^2$. Hence, by the compound probability theorem, if the assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F} = \{A|H, B|AH, AB|H\}$ is coherent, then $z = xy$; that is, $l = u = xy$. In agreement with Remark 2, we observe that $T_L(x, y) \leq xy \leq S_0^H(x, y)$. More in general, given a family $\mathcal{F} = \{A_1|H, A_2|A_1H, \ldots, A_n|A_1 \cdots A_{n-1}H\}$, by iteratively exploiting the associative property, we have

$$\mathcal{C}(\mathcal{F}) = \mathcal{C}(\mathcal{C}(A_1|H, A_2|A_1H), A_3|A_2A_1H, \ldots, A_n|A_1 \cdots A_{n-1}H) =$$

$$= \mathcal{C}(A_1A_2|H, A_3|A_2A_1H, \ldots, A_n|A_1 \cdots A_{n-1}H) = \cdots = A_1A_2 \cdots A_n|H\,;$$

thus, by the compound probability theorem, if the assessment $\mathcal{P} = (p_1, \ldots, p_n, z)$ on $\mathcal{F} \cup \{\mathcal{C}(\mathcal{F})\}$ is coherent, then $z = l = u = p_1 \cdot p_2 \cdots p_n$.

4 Lower and Upper Bounds for the Quasi Conjunction of n Conditional Events

Given the family $\mathcal{F}_n = \{E_1|H_1, \ldots, E_n|H_n\}$, we denote by $\mathcal{C}(\mathcal{F}_n)$ the quasi conjunction of the conditional events in $\mathcal{F}_n$. By the associative property of quasi conjunction, defining $\mathcal{F}_k = \{E_1|H_1, \ldots, E_k|H_k\}$, for each $k = 2, \ldots, n$, it is $\mathcal{C}(\mathcal{F}_k) = \mathcal{C}(\mathcal{C}(\mathcal{F}_{k-1}), E_k|H_k)$. Then, we have

Theorem 2. *Given a probability assessment $\mathcal{P}_n = (p_1, p_2, \ldots, p_n)$ on $\mathcal{F}_n = \{E_1|H_1, \ldots, E_n|H_n\}$, let $[l_k, u_k]$ be the interval of coherent extensions of the assessment $\mathcal{P}_k = (p_1, p_2, \ldots, p_k)$ on the quasi conjunction $\mathcal{C}(\mathcal{F}_k)$, where $\mathcal{F}_k = \{E_1|H_1, \ldots, E_k|H_k\}$. Then, assuming $E_1, H_1, \ldots, E_n, H_n$ logically independent, for each $k = 2, \ldots, n$, we have*

$$l_k = T_L(p_1, p_2, \ldots, p_k)\,, \quad u_k = S_0^H(p_1, p_2, \ldots, p_k)\,,$$

where T_L is the Lukasiewicz t-norm and S_0^H is the Hamacher t-conorm, with parameter $\lambda = 0$.

4.1 The Case $E_1|H_1 \subseteq E_2|H_2 \subseteq \ldots \subseteq E_n|H_n$

In this subsection we give a result on quasi conjunctions when $E_i|H_i \subseteq E_{i+1}|H_{i+1}, i = 1, \ldots, n-1$. We have

Theorem 3. *Given a family* $\mathcal{F}_n = \{E_1|H_1, \ldots, E_n|H_n\}$ *of conditional events such that* $E_1|H_1 \subseteq E_2|H_2 \subseteq \ldots \subseteq E_n|H_n$*, and a coherent probability assessment* $\mathcal{P}_n = (p_1, p_2, \ldots, p_n)$ *on* $\mathcal{F}_n$*, let* $\mathcal{C}(\mathcal{F}_k)$ *be the quasi conjunction of* $\mathcal{F}_k = \{E_i|H_i, i = 1, \ldots, k\}$*,* $k = 2, \ldots, n$*. Moreover, let* $[l_k, u_k]$ *be the interval of coherent extensions on* $\mathcal{C}(\mathcal{F}_k)$ *of the assessment* $(p_1, p_2, \ldots, p_k)$ *on* $\mathcal{F}_k$*. We have: (i)* $E_1|H_1 \subseteq \mathcal{C}(\mathcal{F}_2) \subseteq \ldots \subseteq \mathcal{C}(\mathcal{F}_n) \subseteq E_n|H_n$*; (ii) by assuming no further logical relations, any probability assessment* $(z_2, \ldots, z_k)$ *on* $\{\mathcal{C}(\mathcal{F}_2), \ldots, \mathcal{C}(\mathcal{F}_k)\}$ *is a coherent extension of the assessment* $(p_1, p_2, \ldots, p_k)$ *on* $\mathcal{F}_k$ *if and only if* $p_1 \leq z_2 \leq \cdots \leq z_k \leq p_k$ *,* $k = 2, \ldots, n$*; moreover*

$$l_k = \min(p_1, \ldots, p_k) = p_1 \, , \; u_k = \max(p_1, \ldots, p_k) = p_k \, , \; k = 2, \ldots, n \, .$$

Proof. (i) By iteratively applying (1) and by the associative property of quasi conjunction, we have $\mathcal{C}(\mathcal{F}_{k-1}) \subseteq \mathcal{C}(\mathcal{F}_k) \subseteq E_k|H_k$, $k = 2, \ldots, n$;
(ii) by exploiting the logical relations in point (i), the assertions immediately follow by applying a reasoning similar to that in Remark 3. □

5 Further Aspects on the Lower and Upper Bounds

Now, given any coherent assessment (x, y) on $\{A|H, B|K\}$, we examine further probabilistic aspects on the lower and upper bounds, l and u, for the coherent extensions $z = P[\mathcal{C}(A|H, B|K)]$. More precisely, given any number $\gamma \in [0, 1]$, we are interested in finding:
(i) the set $\mathcal{L}_\gamma$ of the coherent assessments (x, y) on $\{A|H, B|K\}$ such that, for each $(x, y) \in \mathcal{L}_\gamma$, one has $l \geq \gamma$;
(ii) the set $\mathcal{U}_\gamma$ of the coherent assessments (x, y) on $\{A|H, B|K\}$ such that, for each $(x, y) \in \mathcal{U}_\gamma$, one has $u \leq \gamma$.
Case (i). Of course, $\mathcal{L}_0 = [0, 1]^2$; hence we can assume $\gamma > 0$. It must be $l = \max\{x + y - 1, 0\} \geq \gamma$, i.e., $x + y \geq 1 + \gamma$ (as $\gamma > 0$); hence $\mathcal{L}_\gamma$ coincides with the triangle having the vertices $(1, 1), (1, \gamma), (\gamma, 1)$; that is

$$\mathcal{L}_\gamma = \{(x, y) : \gamma \leq x \leq 1, \, 1 + \gamma - x \leq y \leq 1\} \, .$$

Notice that $\mathcal{L}_1 = \{(1, 1)\}$; moreover, for $\gamma \in (0, 1)$, $(\gamma, \gamma) \notin \mathcal{L}_\gamma$.
Case (ii). Of course, $\mathcal{U}_1 = [0, 1]^2$; hence we can assume $\gamma < 1$. We recall that $u = \frac{x+y-2xy}{1-xy}$; hence

$$u - x = \frac{y(1-x)^2}{1-xy} \geq 0 \, , \;\; u - y = \frac{x(1-y)^2}{1-xy} \geq 0 \, ; \tag{2}$$

then, from $u \leq \gamma$ it follows $x \leq \gamma, y \leq \gamma$; hence $\mathcal{U}_\gamma \subseteq [0, \gamma]^2$. Then, taking into account that $x \leq \gamma$ and hence $1 - (2 - \gamma)x > 0$, we have

$$\frac{x + y - 2xy}{1 - xy} \leq \gamma \iff y \leq \frac{\gamma - x}{1 - (2 - \gamma)x} \, ; \tag{3}$$

therefore

$$\mathcal{U}_\gamma = \left\{ (x, y) : \; 0 \leq x \leq \gamma \, , \;\; y \leq \frac{\gamma - x}{1 - (2 - \gamma)x} \right\} \, .$$

Notice that $\mathcal{U}_0 = \{(0,0)\}$; moreover, for $x = y = \gamma \in (0,1)$, it is $u = \frac{2\gamma}{1+\gamma} > \gamma$; hence, for $\gamma \in (0,1)$, $\mathcal{U}_\gamma$ is a strict subset of $[0,\gamma]^2$.
Of course, for every $(x,y) \notin \mathcal{L}_\gamma \cup \mathcal{U}_\gamma$, it is $l < \gamma < u$.
In the next result we determine in general the sets $\mathcal{L}_\gamma, \mathcal{U}_\gamma$.

Theorem 4. *Given a coherent assessment $(p_1, p_2, \ldots, p_n)$ on the family $\{E_1|H_1, \ldots, E_n|H_n\}$, where $E_1, H_1, \ldots, E_n, H_n$ are logically independent, we have*

$$\mathcal{L}_\gamma = \{(p_1, \ldots, p_n) \in [0,1]^n : p_1 + \cdots + p_n \geq \gamma + n - 1\}\,,\ \gamma > 0\,,$$

$$\mathcal{U}_\gamma = \{(p_1, \ldots, p_n) \in [0,1]^n : 0 \leq p_1 \leq \gamma\,,\ \ p_{k+1} \leq r_k\,,\ k = 1, \ldots, n-1\}\,,\ \gamma < 1\,, \tag{4}$$

where $r_k = \frac{\gamma - u_k}{1-(2-\gamma)u_k}$, $u_k = S_0^H(p_1, \ldots, p_k)$, with $\mathcal{L}_0 = \mathcal{U}_1 = [0,1]^n$.

6 QAND Rule and Probabilistic Entailment

We recall that in [4], based on a three-valued calculus of conditional objects, a logic for nonmonotonic reasoning has been proposed. Conditional objects can be seen as the counterpart of the conditional assertions considered in [9] and, for what concerns logical operations, we can look at them as conditional events. Given a set of conditional objects $\mathcal{K}$, we denote by $\mathcal{C}(\mathcal{K})$ the quasi conjunction of the conditional objects in $\mathcal{K}$. In [4] the following inference rule, named QAND, derivable by applying the inference rules of System P (see [9]), has been introduced

$$\text{(QAND)} \qquad\qquad \mathcal{K} \ \Rightarrow\ \mathcal{C}(\mathcal{K})\,.$$

As shown in Section 2, the notions of p-consistency and p-entailment of Adams can be suitable defined in the setting of coherence (see [5, 6]). In the next theorem, to avoid a specific analysis of trivial cases, we assume $\Pi_x = \Pi_y = [0,1]$. We have

Theorem 5. *Given two conditional events $A|H, B|K$, with $\Pi_x = \Pi_y = [0,1]$, we have*

$$A|H \ \Rightarrow_p\ B|K \iff A|H \ \subseteq\ B|K\,.$$

The next result, related to the approach of Adams, deepens in the framework of coherence the probabilistic semantics of the QAND rule.

Theorem 6. *Given a p-consistent family $\mathcal{F}_n = \{E_i|H_i, i = 1, \ldots, n\}$ and denoting by $\mathcal{C}(\mathcal{F}_n)$ the associated quasi conjunction, for every $\varepsilon \in (0,1]$ there exist $\delta_1 \in (0,1], \ldots, \delta_n \in (0,1]$ such that, for every coherent assessment $(p_1, \ldots, p_n, z)$ on $\mathcal{F}_n \cup \{\mathcal{C}(\mathcal{F}_n)\}$, where $p_i = P(E_i|H_i)$, $z = P(\mathcal{C}(\mathcal{F}_n))$, if $p_1 \geq 1-\delta_1, \ldots, p_n \geq 1-\delta_n$, then $z \geq 1-\varepsilon$. Hence, we have $\mathcal{F}_n \ \Rightarrow_p\ \mathcal{C}(\mathcal{F}_n)$.*

Recalling Remark 1, in the next result we show that p-entailment of a conditional event $B|A$ from a family $\mathcal{F}_n$ is equivalent to the existence of a non-empty subset $\mathcal{S}$ of $\mathcal{F}_n$ such that $\mathcal{C}(\mathcal{S})$ p-entails $B|A$.

Theorem 7. *A p-consistent family of conditional events $\mathcal{F}_n$ p-entails a conditional event $B|A$ if and only if there exists a non-empty subset $\mathcal{S}$ of $\mathcal{F}_n$ such that $\mathcal{C}(\mathcal{S})$ p-entails $B|A$.*

An example. We illustrate Theorem 7 by using the well known inference rules Cautious Monotonicity (CM), Or, and Cut, as shown below.
(CM) If $\{C|A, B|A\} \subseteq \mathcal{F}_n$, then $\mathcal{F}_n \Rightarrow_p C|AB$. The assertion follows by observing that, defining $\mathcal{S} = \{C|A, B|A\}$, it is $\mathcal{C}(\mathcal{S}) = BC|A \subseteq C|AB$, so that $\mathcal{C}(\mathcal{S}) \Rightarrow_p C|AB$.
(Or) If $\{C|A, C|B\} \subseteq \mathcal{F}_n$, then $\mathcal{F}_n \Rightarrow_p C|(A \vee B)$. The assertion follows by observing that, defining $\mathcal{S} = \{C|A, C|B\}$, it is $\mathcal{C}(\mathcal{S}) = C|(A \vee B)$, so that, trivially, $\mathcal{C}(\mathcal{S}) \Rightarrow_p C|(A \vee B)$.
(Cut) If $\{C|AB, B|A\} \subseteq \mathcal{F}_n$, then $\mathcal{F}_n \Rightarrow_p C|A$. The assertion follows by observing that, defining $\mathcal{S} = \{C|AB, B|A\}$, it is $\mathcal{C}(\mathcal{S}) = BC|A \subseteq C|A$, so that $\mathcal{C}(\mathcal{S}) \Rightarrow_p C|A$.
Of course, in the previous inference rules, the entailment of the conclusion from $\mathcal{F}_n$ also follows by directly applying Definition 2, as made in [5].

7 Conclusions

We have studied, in a coherence-based setting, the extensions of a given probability assessment on n conditional events to their quasi conjunction, by also considering two cases of logical dependency. We have analyzed further probabilistic aspects on quasi conjunction, by also examining the relation between the notion of p-entailment and the inclusion relation of Goodman and Nguyen. Then, we have shown that each p-consistent family $\mathcal{F}$ p-entails the quasi conjunction $\mathcal{C}(\mathcal{F})$. Finally, we have given a result on the equivalence between p-entailment from $\mathcal{F}$ and p-entailment from $\mathcal{C}(\mathcal{S})$, where $\mathcal{S}$ is some non-empty subset of $\mathcal{F}$.

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