Lower and Upper Probability Bounds for Some Conjunctions of Two Conditional Events

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Abstract. In this paper we consider, in the framework of coherence, four different definitions of conjunction among conditional events. In each of these definitions the conjunction is still a conditional event. We first recall the different definitions of conjunction; then, given a coherent probability assessment (x, y) on a family of two conditional events $\{A|H, B|K\}$, for each conjunction $(A|H) \wedge (B|K)$ we determine the (best) lower and upper bounds for the extension $z = P[(A|H) \wedge (B|K)]$. We show that, in general, these lower and upper bounds differ from the classical Fréchet-Hoeffding bounds. Moreover, we recall a notion of conjunction studied in recent papers, such that the result of conjunction of two conditional events $A|H$ and $B|K$ is (not a conditional event, but) a suitable conditional random quantity, with values in the interval $[0, 1]$. Then, we remark that for this conjunction, among other properties, the Fréchet-Hoeffding bounds are preserved.

Keywords: Coherence · Conditional event
Conditional random quantity
Kleene-Lukasiewicz-Heyting conjunction · Lukasiewicz conjunction
Bochvar internal conjunction · Sobocinski conjunction
Lower and upper bounds · Fréchet-Hoeffding bounds

1 Introduction

In probability theory and in probability logic a relevant problem, largely discussed by many authors (see, e.g., [2,8,9,26]), is that of suitably defining logical operations among conditional events. In this paper we consider four different notions of conjunction among conditional events such that in all cases the result of conjunction is a conditional event too: Kleene-Lukasiewicz-Heyting conjunction, Lukasiewicz conjunction, Bochvar internal conjunction, Sobocinski conjunction. For each conjunction $(A|H) \wedge (B|K)$, given the conditional probabilities

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$x = P(A|H)$ and $y = P(B|K)$, we determine the (best) lower and upper bounds for the conditional probability $z = P[(A|H) \wedge (B|K)]$, that is, given a coherent assessment (x, y) on the family $\{A|H, B|K\}$, we determine the lower and upper bounds z', z'' such that the extension (x, y, z) on $\{A|H, B|K, (A|H) \wedge (B|K)\}$, with $z = P[(A|H) \wedge (B|K)]$, is coherent if and only if $z \in [z', z'']$. Of course, $z', z'' \in [0, 1]$, but the extension $(x, y, 0)$ (resp., $(x, y, 1)$) is coherent if only if $z' = 0$ (resp., $z'' = 1$). We verify that in all cases such probability bounds do not coincide with the classical Fréchet-Hoeffding bounds: $z' = \max\{x + y - 1, 0\}$ and $z'' = \min\{x, y\}$. In particular, we obtain $z' = 0$ and $z'' = \min\{x, y\}$ for the Kleene-Lukasiewicz-Heyting conjunction and for the Lukasiewicz conjunction. We obtain $z' = 0$ and $z'' = 1$ for the Bochvar internal conjunction. Finally, for the Sobocinski conjunction we obtain $z' = \max\{x + y - 1, 0\}$ and $z'' = S_0^H(x, y)$, where $S_0^H(x, y)$ is the Hamacher t-conorm with parameter $\lambda = 0$. Then, we examine a notion of conjunction introduced in some recent papers, where the result of conjunction in general is not a conditional event, but a conditional random quantity. We remark that this notion of conjunction, differently from the previous of notions of conjunction, satisfy many properties. In particular the classical Fréchet-Hoeffding bounds are satisfied. We also recall a dual notion of disjunction by showing that within this approach the *prevision sum rule* is satisfied, that is $\mathbb{P}[(A|H) \vee (B|K)] = \mathbb{P}(A|H) + \mathbb{P}(B|K) - \mathbb{P}[(A|H) \wedge (B|K)]$.

2 Preliminary Notions and Results

An event A is a (non ambiguous) logical proposition which describes an uncertain fact; hence A is a two-valued logical entity which can be true (T), or false (F). The indicator of A, denoted by the same symbol, is a two-valued numerical quantity which is 1, or 0, according to whether the event A is true, or false, respectively. The sure event is denoted by Ω and the impossible event is denoted by $\emptyset$. Moreover, we denote by $A \wedge B$ the logical conjunction and by $A \vee B$ the logical disjunction. In many cases the conjunction between A and B is simple denoted as the product AB. The negation of an event A is denoted by $\overline{A}$. Given two events A and B, the inclusion relation between A and B, that is $A\overline{B} = \emptyset$, is denoted by $A \subseteq B$. We recall that n events are logically independent when the number of atoms, or constituents, generated by them is 2^n. In case of some logical dependencies among the events, the number of atoms is less than 2^n. Given two events A and B, with $A \neq \emptyset$, the conditional event $B|A$ is looked at as a three-valued logical entity which is true (T), or false (F), or void (V), according to whether AB is true, or $A\overline{B}$ is true, or $\overline{A}$ is true.

Coherent Conditional Probability Assessments. We recall that, using the betting scheme of de Finetti [12], if you assess $P(B|A) = p$, then you agree to pay an amount p, by receiving 1, or 0, or p, according to whether AB is true, or $A\overline{B}$ is true, or $\overline{A}$ is true (bet called off). Then, the random gain associated with the assessment $P(B|A) = p$ is $G = sH(E - p)$, where s is a non zero real number. More in general, let be given a real function $P : \mathcal{F} \rightarrow \mathbb{R}$, where $\mathcal{K}$ is an arbitrary family of conditional events. Given any subfamily $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathcal{K}$, the restriction of P to $\mathcal{F}$ is the vector $\mathcal{P}_n = (p_1, \ldots, p_n)$, where $p_i = P(E_i|H_i)\,, \quad i = 1, \ldots, n$. We denote by $\mathcal{H}_n$ the disjunction $H_1 \vee \cdots \vee H_n$. As $E_iH_i \vee \overline{E}_iH_i \vee \overline{H}_i = \Omega\,, \quad i = 1, \ldots, n$, by expanding the expression $\bigwedge_{i=1}^{n}(E_iH_i \vee \overline{E}_iH_i \vee \overline{H}_i)$, we can represent Ω as the disjunction of 3^n logical conjunctions, some of which may be impossible. The remaining ones are the atoms, or constituents, generated by the family $\mathcal{F}$ and, of course, are a partition of Ω. We denote by $C_1, \ldots, C_m$ the constituents contained in $\mathcal{H}_n$ and (if $\mathcal{H}_n \neq \Omega$) by C_0 the remaining constituent $\overline{\mathcal{H}}_n = \overline{H}_1 \cdots \overline{H}_n$, so that

$$\mathcal{H}_n = C_1 \vee \cdots \vee C_m\,, \quad \Omega = \overline{\mathcal{H}}_n \vee \mathcal{H}_n = C_0 \vee C_1 \vee \cdots \vee C_m\,, \quad m + 1 \leq 3^n\,.$$

In the betting metaphor, $G = \sum_{i=1}^{n} s_iH_i(E_i - p_i)$ is the random gain associated with $(\mathcal{F}, \mathcal{P})$, where $s_1, \ldots, s_n$ are n arbitrary real numbers, which is the difference between the amount that you receive, $\sum_{i=1}^{n} s_i(E_iH_i + p_i\overline{H}_i)$, and the amount that you pay, $\sum_{i=1}^{n} s_ip_i$. Let g_h be the value of G when C_h is true; of course $g_0 = 0$. Denoting by $\mathcal{G}_{\mathcal{H}_n}$ the set of possible values of G restricted to $\mathcal{H}_n$, it is $\mathcal{G}_{\mathcal{H}_n} = \{g_1, \ldots, g_m\}$. Then, we have

Definition 1. The function P defined on an arbitrary family of conditional events $\mathcal{K}$ is coherent if and only if, for every finite subfamily $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ of $\mathcal{K}$, one has: $\min\ \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \max\ \mathcal{G}_{\mathcal{H}_n}$.

As shown by Definition 1, a probability assessment is coherent if and only if, in any finite combination of n bets, it may not happen that the values $g_1, \ldots, g_m$ are all positive, or all negative (*no Dutch Book*).

Given any integer n we set $J_n = \{1, 2, \ldots, n\}$; for each $h \in J_m$ with the constituent C_h we associate a point $Q_h = (q_{h1}, \ldots, q_{hn})$, where $q_{hj} = 1$, or 0, or p_j, according to whether $C_h \subseteq E_jH_j$, or $C_h \subseteq \overline{E}_jH_j$, or $C_h \subseteq \overline{H}_j$.

Denoting by $\mathcal{I}$ the convex hull of $Q_1, \ldots, Q_m$, by a suitable alternative theorem [13, Theorem 2.9], the condition $\mathcal{P} \in \mathcal{I}$ is equivalent to the condition $\min \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \max \mathcal{G}_{\mathcal{H}_n}$ given in Definition 1 (see, e.g., [17,21]). Moreover, the condition $\mathcal{P} \in \mathcal{I}$ amounts to the solvability of the following system (Σ) in the unknowns $\lambda_1, \ldots, \lambda_m$

$$(\Sigma): \qquad \sum_{h=1}^{m} q_{hj}\lambda_h = p_j\,, \quad j \in J_n\,; \ \ \sum_{h=1}^{m} \lambda_h = 1\,; \ \ \lambda_h \geq 0\,,\ h \in J_m\,. \tag{1}$$

We say that system (Σ) is associated with the pair $(\mathcal{F}, \mathcal{P})$. Hence, the following result provides a characterization of the notion of coherence given in Definition 1 ([14, Theorem 4.4], see also [15,20,21])

Theorem 1. *The function P defined on an arbitrary family of conditional events $\mathcal{K}$ is coherent if and only if, for every finite subfamily $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ of $\mathcal{K}$, denoting by $\mathcal{P}$ the vector $(p_1, \ldots, p_n)$, where $p_j = P(E_j|H_j)$, $j = 1, 2, \ldots, n$, the system (Σ) associated with the pair $(\mathcal{F}, \mathcal{P})$ is solvable.*

Coherence Checking. We recall now some results on the coherence checking of a probability assessment on a finite family of conditional events. Given a probability assessment $\mathcal{P} = (p_1, \dots, p_n)$ on $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$, let S be the set of solutions $\Lambda = (\lambda_1, \dots, \lambda_m)$ of the system (Σ). Then, assuming $S \neq \emptyset$, we define

$$\begin{array}{l}\Phi_j(\Lambda) = \Phi_j(\lambda_1, \dots, \lambda_m) = \sum_{r:C_r \subseteq H_j} \lambda_r \,, \quad j \in J_n \,;\; \Lambda \in S \,; \\ M_j = \max_{\Lambda \in S} \; \Phi_j(\Lambda) \,, \quad j \in J_n \,; \quad I_0 = \{j \,:\, M_j = 0\} \,.\end{array}$$

If $S \neq \emptyset$, then S is a closed bounded set and the maximum M_j of the linear function $\Phi_j(\Lambda) = \sum_{r:C_r \subseteq H_j} \lambda_r$ there exists for every $j \in J_n$. Assuming $\mathcal{P}$ coherent, each solution $\Lambda = (\lambda_1, \dots, \lambda_m)$ of system (Σ) is a coherent extension of the assessment $\mathcal{P}$ on $\mathcal{F}$ to the family $\{C_1|\mathcal{H}_n, C_2|\mathcal{H}_n, \dots, C_m|\mathcal{H}_n\}$. Then, for each solution Λ of system (Σ) the quantity $\Phi_j(\Lambda)$ is the conditional probability $P(H_j|\mathcal{H}_n)$. Moreover, the quantity M_j is the upper probability $P''(H_j|\mathcal{H}_n)$ over all the solutions Λ of system (Σ). Of course, $j \in I_0$ if and only if $P''(H_j|\mathcal{H}_n) = 0$. Notice that I_0 is a strict subset of J_n. If I_0 is nonempty, we set $\mathcal{F}_0 = \{E_i|H_i \in \mathcal{F} : i \in I_0\}$ and $\mathcal{P}_0 = (P(E_i|H_i), i \in \mathcal{I}_0)$. We say that the pair $(\mathcal{F}_0, \mathcal{P}_0)$ is associated with I_0. Then, we have [16, Theorem 3.3].

Theorem 2. *The assessment $\mathcal{P}$ on $\mathcal{F}$ is coherent if and only if the following conditions are satisfied: (i) the system (Σ) associated with the pair $(\mathcal{F}, \mathcal{P})$ is solvable; (ii) if $I_0 \neq \emptyset$, then $\mathcal{P}_0$ is coherent.*

By Theorem 2, the following algorithm checks in a finite number of steps the coherence of the probability assessment $\mathcal{P}$ on a finite family of conditional events $\mathcal{F}$.

Algorithm 1. Let be given the pair $(\mathcal{F}, \mathcal{P})$.

1. Construct the system (Σ) associated with the pair $(\mathcal{F}, \mathcal{P})$ and check its solvability;
2. If the system (Σ) is not solvable then $\mathcal{P}$ is not coherent and the procedure stops, otherwise compute the set I_0;
3. If $I_0 = \emptyset$ then $\mathcal{P}$ is coherent and the procedure stops, otherwise set $(\mathcal{F}, \mathcal{P}) = (\mathcal{F}_0, P_0)$ and go to step 1.

In the next definition we recall the notion of inclusion relation between two conditional events (see, e.g., [25]).

Definition 2. *Given two conditional events $A|H$ and $B|K$ we define that $A|H$* logically implies *$B|K$, which we denote by $A|H \subseteq B|K$, if and only if AH is* true *implies BK is* true *and $\overline{B}K$ is* true *implies $\overline{A}H$ is* true*; i.e., $AH \subseteq BK$ and $\overline{B}K \subseteq \overline{A}H$.*

By coherence, it holds that (see, e.g., [19, Theorem 1]) $P(A|H) \leq P(B|K)$ when $A|H \subseteq B|K$. It can be also verified that [21, Remark 3]

$$A|H \subseteq B|K \iff AH\overline{B}K = \overline{H}\,\overline{B}K = AH\overline{K} = \emptyset \,. \tag{2}$$

We recall that the notion of implication given in [11] introduces a suitable conditional event associated with $A|H$ and $B|K$ (see also [27,28]). Between the notions of implication and inclusion relation there exists a relationship: as shown by (2), when the inclusion relation holds the implication between $A|H$ and $B|K$ is void or true. It is void if $\overline{H}\,\overline{K}$ is true; it is true in all the other cases.

3 Computation of Lower and Upper Bounds for Different Notions of the Conjunction

In this section we examine four different notions of conjunction between three valued events ([4], see also [5]), named in our approach conditional events: Kleene-Lukasiewicz-Heyting conjunction ($\wedge_K$), Lukasiewicz conjunction ($\wedge_L$), Bochvar internal conjunction, also known as Kleene weak conjunction ($\wedge_B$), and Sobocinski conjunction ($\wedge_S$). In all these definitions the conjunction of two conditional events is still a conditional event. We observe that, differently from other definitions of conjunctions, this four logical operations are all commutative and associative. The truth values of the four conjunctions are given in Table 1. We recall that a conditional event $A|H$, where A, H are two events with $H \neq \emptyset$, can be looked as a three valued random quantity $A|H = AH + x\overline{H} \in \{1, 0, x\}$, where $x = P(A|H)$. Let A, H, B, K be logical independent events, with $H \neq \emptyset, K \neq \emptyset$. Assuming that $P(A|H) = x$, $P(B|K) = y$ and $P[(A|H) \wedge (B|K)] = z$ the table of logical values for the different notions of conjunction are given in Table 2. We list below in an explicit way the four conjunctions as conditional events.

1. $(A|H) \wedge_K (B|K) = AHBK|(HK \vee \overline{A}H \vee \overline{B}K)$;
2. $(A|H) \wedge_L (B|K) = AHBK|(HK \vee \overline{A}\,\overline{B} \vee \overline{A}\,\overline{K} \vee \overline{B}\,\overline{H} \vee \overline{H}\,\overline{K})$;
3. $(A|H) \wedge_B (B|K) = AHBK|HK$;
4. $(A|H) \wedge_S (B|K) = (AH \vee \overline{H}) \wedge (BK \vee \overline{K})|(H \vee K)$.

Table 1. Truth values of the conjunctions. The values T, F, V denote *True*, *False*, and *Void*, respectively.

	C_h	$A\|H$	$B\|K$	$\wedge_K$	$\wedge_L$	$\wedge_B$	$\wedge_S$
C_1	$AHBK$	T	T	T	T	T	T
C_2	$AH\overline{B}K$	T	F	F	F	F	F
C_3	$AH\overline{K}$	T	V	V	V	V	T
C_4	$\overline{A}HBK$	F	T	F	F	F	F
C_5	$\overline{A}H\overline{B}K$	F	F	F	F	F	F
C_6	$\overline{A}H\,\overline{K}$	F	V	F	F	V	F
C_7	$\overline{H}BK$	V	T	V	V	V	T
C_8	$\overline{H}\,\overline{B}K$	V	F	F	F	V	F
C_0	$\overline{H}\,\overline{K}$	V	V	V	F	V	V

Table 2. Numerical values of the conjunctions. The values x, y, z denote $P(A|H)$, $P(B|K)$ and $P[(A|H) \wedge (B|K)]$, respectively.

	C_h	$A\|H$	$B\|K$	$\wedge_K$	$\wedge_L$	$\wedge_B$	$\wedge_S$
C_1	$AHBK$	1	1	1	1	1	1
C_2	$AH\overline{B}K$	1	0	0	0	0	0
C_3	$AH\overline{K}$	1	y	z	z	z	1
C_4	$\overline{A}HBK$	0	1	0	0	0	0
C_5	$\overline{A}H\overline{B}K$	0	0	0	0	0	0
C_6	$\overline{A}H\,\overline{K}$	0	y	0	0	z	0
C_7	$\overline{H}BK$	x	1	z	z	z	1
C_8	$\overline{H}\,\overline{B}K$	x	0	0	0	z	0
C_0	$\overline{H}\,\overline{K}$	x	y	z	0	z	z

3.1 The Kleene-Lukasiewicz-Heyting Conjunction

The Kleene-Lukasiewicz-Heyting conjunction is represented in Table 1 by the symbol $\wedge_K$. This notion coincides with the logical product between tri-events given in [11] (see also [27]). As shown in Table 2, based on the betting scheme the conjunction of two conditional events $A|H$ and $B|K$ in our approach coincides with the random quantity

$$(A|H) \wedge_K (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}H \vee \overline{B}K \text{ is true,} \\ z, & \text{if } AH\overline{K} \vee \overline{H}BK \vee \overline{H}\,\overline{K} \text{ is true,} \end{cases} \tag{3}$$

where $z = P(A|H \wedge_K B|K)$. Then,

$$(A|H) \wedge_K (B|K) = 1 \cdot AHBK + z(\overline{H}BK \vee AH\overline{K} \vee \overline{H}\,\overline{K})\,. \tag{4}$$

Notice that the quantity $z = \mathbb{P}[(A|H) \wedge_K (B|K)]$ represents the value that you assess, with the proviso that, you will pay the amount sz by receiving the random quantity $s[(A|H) \wedge_K (B|K)]$. In particular, if $s = 1$, then you agree to pay z with the proviso that you will receive: 1, if both conditional events are true; 0, if at least one of the conditional events is false; z, otherwise. Based on (3) and (4), the Kleene-Lukasiewicz-Heyting conjunction is the following conditional event

$$(A|H) \wedge_K (B|K) = AHBK|(AHBK \vee \overline{A}H \vee \overline{B}K). \tag{5}$$

Remark 1. We observe that $(A|H) \wedge_K (B|K) \subseteq A|H$ and $(A|H) \wedge_K (B|K) \subseteq B|K$. Then, coherence requires that

$$P[(A|H) \wedge_K (B|K)] \leq \min\{P(A|H), P(B|K)\}.$$

Moreover, Table 1 shows that $(A|H) \wedge_K (B|K)$ and $A|H$ only differ when $AH\overline{B}K$ is true, or $\overline{H}\,\overline{B}K$ is true, or $AH\overline{K}$ is true. Then, from (2) and Table 1 we recover the definition of inclusion relation in the form given in [25], that is

$$A|H \subseteq B|K \Longleftrightarrow (A|H) \wedge_K (B|K) = A|H.$$

Remark 2. We recall that given two conditional events $\{A|H, B|K\}$, with A, H, B, K logically independent, and with $H \neq \emptyset, K \neq \emptyset$, the set of all coherent assessments (x, y) on $\{A|H, B|K\}$ is the unit square $[0, 1]^2$.

In the next result we give the values of $z = P[(A|H) \wedge_K (B|K)]$ which are coherent extensions of a probability assessment (x, y) on $\{A|H, B|K\}$.

Theorem 3. *Given any coherent assessment (x, y) on $\{A|H, B|K\}$, with A, H, B, K logically independent, and with $H \neq \emptyset, K \neq \emptyset$, the probability assessment $z = P[(A|H) \wedge_K (B|K)]$ is a coherent extension if and only if $z \in [z', z'']$, where $z' = 0$ and $z'' = \min\{x, y\}$.*

Proof. We recall that, by Remark 2, any assessment $(x, y) \in [0, 1]^2$ is coherent. The constituents C_h's and the points Q_h's associated with $(\mathcal{F}, \mathcal{P})$, where $\mathcal{F} = \{A|H,\ B|K,\ (A|H) \wedge_K (B|K)\}$, $\mathcal{P} = (x, y, z)$, are given in Table 3. The constituents C_h's contained in $\mathcal{H}_3 = H \vee K$ are $C_1, \dots, C_8$. We recall that coherence of the probability assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F}$ requires that the condition $\mathcal{P} \in \mathcal{I}$ be satisfied, where $\mathcal{I}$ is the convex hull of $Q_1, \dots, Q_8$. This amounts to the solvability of the following system

$$(\Sigma) \qquad \mathcal{P} = \textstyle\sum_{h=1}^{8} \lambda_h Q_h,\ \sum_{h=1}^{8} \lambda_h = 1,\ \lambda_h \geq 0,\ h = 1, \dots, 8;$$

that is

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 + x\lambda_7 + x\lambda_8 = x,\ \lambda_1 + y\lambda_3 + \lambda_4 + y\lambda_6 + \lambda_7 = y \\ \lambda_1 + z\lambda_3 + z\lambda_7 = z,\ \ \sum_{h=1}^{8} \lambda_h = 1,\ \lambda_h \geq 0,\ h = 1, \dots, 8\,. \end{cases}$$

We first prove that $z' = 0$, by verifying that the assessment $(x, y, 0)$ is coherent for any pair $(x, y) \in [0, 1]^2$. We distinguish the following cases: (i) $x < 1, y < 1$; (ii) $x = 1,\ y < 1$; (iii) $x < 1,\ y = 1$; (iv) $x = y = 1$.
Case (i). We observe that $(x, y, 0) \in \mathcal{I}$. Indeed, the system (Σ) is solvable, with a solution given by

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = \tfrac{x(1-y)}{1-xy}, \lambda_4 = 0, \lambda_5 = \tfrac{(1-x)(1-y)}{1-xy}, \lambda_6 = 0, \lambda_7 = \tfrac{y(1-x)}{1-xy}, \lambda_8 = 0,$$

that is

$$(x, y, 0) = \frac{x(1-y)}{1-xy} Q_3 + \frac{(1-x)(1-y)}{1-xy} Q_5 + \frac{y(1-x)}{1-xy} Q_7.$$

Moreover, this solution is such that $\lambda_5 = \frac{(1-x)(1-y)}{1-xy} > 0$, then

$$\begin{aligned}
&\textstyle\sum_{h: C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = \frac{x(1-y)}{1-xy} + \frac{(1-x)(1-y)}{1-xy} = \frac{1-y}{1-xy} > 0, \\
&\textstyle\sum_{h: C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 = \frac{(1-x)(1-y)}{1-xy} + \frac{y(1-x)}{1-xy} = \frac{1-x}{1-xy} > 0, \\
&\textstyle\sum_{h: C_h \subseteq AHBK \vee \overline{A}H \vee \overline{B}K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8 = \lambda_5 = \frac{(1-x)(1-y)}{1-xy} > 0.
\end{aligned}$$

Table 3. Constituents C_h's and points Q_h's associated with the prevision assessment $\mathcal{P}=(x,y,z)$ on $\mathcal{F}=\{A|H, B|K, (A|H)\wedge_K (B|K)\}$.

	C_h	Q_h	
C_1	$AHBK$	(1,1,1)	Q_1
C_2	$AH\overline{B}K$	$(1,0,0)$	Q_2
C_3	$AH\overline{K}$	$(1,y,z)$	Q_3
C_4	$\overline{A}HBK$	$(0,1,0)$	Q_4
C_5	$\overline{A}H\overline{B}K$	$(0,0,0)$	Q_5
C_6	$\overline{A}H\overline{K}$	$(0,y,0)$	Q_6
C_7	$\overline{H}BK$	$(x,1,z)$	Q_7
C_8	$\overline{H}\,\overline{B}K$	$(x,0,0)$	Q_8
C_0	$\overline{H}\,\overline{K}$	(x,y,z)	$Q_0=\mathcal{P}$

Thus, $I_0=\emptyset$ and from Theorem 2 it follows that $(x,y,0)$ is coherent.

Case (ii). It holds that $\mathcal{P}=(1,y,0)=\frac{1}{2}(1,y,0)+\frac{y}{2}(1,1,0)+\frac{(1-y)}{2}(1,0,0)=\frac{1}{2}Q_3+\frac{y}{2}Q_7+\frac{1-y}{2}Q_8$, therefore the vector $(0,0,\frac{1}{2},0,0,0,\frac{y}{2},\frac{1-y}{2})$ is a solution of (Σ) such that $\lambda_8=1-y>0$, $\sum_{h:C_h\subseteq H}\lambda_h=\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6=\frac{1}{2}>0$, $\sum_{h:C_h\subseteq K}\lambda_h=\lambda_1+\lambda_2+\lambda_4+\lambda_5+\lambda_7+\lambda_8=\frac{1}{2}>0$, and $\sum_{h:C_h\subseteq AHBK\vee\overline{A}H\vee\overline{B}K}\lambda_h=\lambda_1+\lambda_2+\lambda_4+\lambda_5+\lambda_6+\lambda_8=\frac{1-y}{2}>0$. Then $I_0=\emptyset$ and from Theorem 2 it follows that $(x,y,0)$ is coherent.

Case (iii). The analysis is similar to the case (ii).

Case (iv). The system (Σ) becomes

$$\lambda_1=0,\ \lambda_2+\lambda_3+\lambda_7+\lambda_8=1,\ \lambda_3+\lambda_4+\lambda_6+\lambda_7=1, \sum_{h=1}^{8}\lambda_h=1,\ \lambda_h\geq 0,\ \forall\, h,$$

or equivalently: $\lambda_3+\lambda_7=1,\ \ \lambda_3\geq 0,\ \ \lambda_7\geq 0, \lambda_1=\lambda_2=\lambda_4=\lambda_5=\lambda_6=\lambda_8=0$. Then, the set S of solutions of (Σ) is $S=\{(0,0,\lambda,0,0,0,1-\lambda,0):0\leq\lambda\leq 1\}$. We observe that $\sum_{h:C_h\subseteq H}\lambda_h=\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6=\lambda$, $\sum_{h:C_h\subseteq K}\lambda_h=\lambda_1+\lambda_2+\lambda_4+\lambda_5+\lambda_7+\lambda_8=1-\lambda$, and $\sum_{h:C_h\subseteq AHBK\vee\overline{A}H\vee\overline{B}K}\lambda_h=\lambda_1+\lambda_2+\lambda_4+\lambda_5+\lambda_6+\lambda_8=0$. For $0<\lambda<1$ it holds that $\sum_{h:C_h\subseteq H}\lambda_h>0$ and $\sum_{h:C_h\subseteq K}\lambda_k>0$. Then, $\mathcal{I}_0=\{3\}$ and by Algorithm 1, from coherence of the assessment $P(AHBK|(AHBK\vee\overline{A}H\vee\overline{B}K))=z=0$, the assessment $(1,1,0)$ is coherent too.

Thus, $(x,y,0)$ is coherent for every $(x,y)\in[0,1]$ and hence $z'=0$.

Concerning the upper bound, by Remark 1 coherence requires that $z\leq\min\{x,y\}$. We prove that $z''=\min\{x,y\}$, by verifying that the assessment $(x,y,\min\{x,y\})$ is coherent for every $(x,y)\in[0,1]^2$. We first show that the point $(x,y,\min\{x,y\})$ is always a linear convex combinations of a subset of the set $\{Q_1,Q_2,Q_4,Q_5\}$. By assuming $\min\{x,y\}=x$, we have

$$(x,y,x)=xQ_1+(y-x)Q_4+(1-y)Q_5=x(1,1,1)+(y-x)(0,1,0)+(1-y)(0,0,0).$$

Then $(x,y,x)\in\mathcal{I}$ and the system (Σ) is solvable, with $\lambda_1=x,\lambda_4=y-x,\lambda_5=1-y,\lambda_2=\lambda_3=\lambda_6=\lambda_7=\lambda_8=0$. Moreover, $\sum_{h:C_h\subseteq H}\lambda_h=\lambda_1+\lambda_2+\lambda_3+$

$\lambda_4 + \lambda_5 + \lambda_6 = 1 > 0$, $\sum_{h:C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 = 1 > 0$, and $\sum_{h:C_h \subseteq AHBK \vee \overline{A}H \vee \overline{B}K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_8 = 1 > 0$. Therefore $I_0 = \emptyset$ and $(x, y, \min\{x, y\})$ is coherent for every $(x, y) \in [0, 1]^2$ such that $x \leq y$. By a similar reasoning $(x, y, \min\{x, y\})$ is coherent for every $(x, y) \in [0, 1]^2$ such that $y < x$.

Thus $(x, y, \min\{x, y\})$ is coherent for every $(x, y) \in [0, 1]^2$ and hence $z'' = \min\{x, y\}$. □

We recall that in [26, p. 161] lower and upper bounds for $(A|H) \wedge_K (B|K)$ have been obtained based on different premises.

3.2 The Lukasiewicz Conjunction

The Lukasiewicz conjunction is represented in Table 1 by the symbol $\wedge_L$. As shown in Table 2, based on the betting scheme the conjunction of two conditional events $A|H$ and $B|K$ in our approach coincides with the random quantity

$$(A|H) \wedge_L (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}H \vee \overline{B}K \vee \overline{H}\,\overline{K} \text{ is true,} \\ z, & \text{if } AH\overline{K} \vee \overline{H}BK \text{ is true,} \end{cases} \tag{6}$$

where $z = P(A|H \wedge_L B|K)$. Then,

$$(A|H) \wedge_L (B|K) = 1 \cdot AHBK + z(AH\overline{K} \vee \overline{H}BK)\,. \tag{7}$$

Based on (6) and (7), by observing that

$$\begin{aligned} &\overline{AH\overline{K} \vee \overline{H}BK} = \\ &= AHBK \vee AH\overline{B}K \vee \overline{H}\,\overline{K} \vee \overline{H}\,\overline{B}K \vee \overline{A}HBK \vee \overline{A}H\overline{K} \vee \overline{A}H\overline{B}K = \\ &= HK \vee \overline{A}\,\overline{B} \vee \overline{A}\,\overline{K} \vee \overline{B}\,\overline{H} \vee \overline{H}\,\overline{K}, \end{aligned}$$

the Lukasiewicz conjunction is the following conditional event

$$(A|H) \wedge_L (B|K) = AHBK|(HK \vee \overline{A}\,\overline{B} \vee \overline{A}\,\overline{K} \vee \overline{B}\,\overline{H} \vee \overline{H}\,\overline{K}). \tag{8}$$

As we can see from (3) and (6) (see also Table 1), the conjunctions $(A|H) \wedge_K (B|K)$ and $(A|H) \wedge_L (B|K)$ only differ when $\overline{H}\,\overline{K}$ is true; indeed in this case $(A|H) \wedge_K (B|K) = z$, where $z = P[(A|H) \wedge_K (B|K)]$, and $(A|H) \wedge_K (B|K) = 0$. Then, by observing that $z \geq 0$, it holds that

$$(A|H) \wedge_K (B|K) \geq (A|H) \wedge_L (B|K) \tag{9}$$

and hence

$$P[(A|H) \wedge_K (B|K)] \geq P[(A|H) \wedge_L (B|K)]. \tag{10}$$

On the other hand, as it can be verified, it holds that $(A|H) \wedge_L (B|K) \subseteq (A|H) \wedge_K (B|K)$, from which it follows (10). Concerning the lower and upper bounds on $(A|H) \wedge_L (B|K)$, we have

Theorem 4. *Given any coherent assessment (x, y) on $\{A|H, B|K\}$, with A, H, B, K logically independent, and with $H \neq \emptyset, K \neq \emptyset$, the probability assessment $z = P[(A|H) \wedge_L (B|K)]$ is a coherent extension if and only if $z \in [z', z'']$, where $z' = 0$ and $z'' = \min\{x, y\}$.*

Proof. By Theorem 3 the lower bound on $P[(A|H) \wedge_K (B|K)]$ is 0; then, from (10) the lower bound z' on $P[(A|H) \wedge_L (B|K)]$ is still 0. Moreover, from (10) it also follows that $z'' \leq \min\{x, y\}$. We will prove that $z'' = \min\{x, y\}$. The constituents C_h's and the points Q_h's associated with $(\mathcal{F}, \mathcal{P})$, where $\mathcal{F} = \{A|H,\ B|K,\ (A|H) \wedge_L (B|K)\}$, $\mathcal{P} = (x, y, z)$, are given in Table 4. We observe that $\mathcal{H}_3 = H \vee K \vee HK \vee \overline{A}\,\overline{B} \vee \overline{A}\,\overline{K} \vee \overline{B}\,\overline{H} \vee \overline{H}\,\overline{K} = \Omega$. Then, the constituents C_h's contained in $\mathcal{H}_3 = \Omega$ are $C_1, \ldots, C_9$, that is all constituents $C_1, \ldots, C_8$ associated with the conjunction $\wedge_K$ plus the constituent $C_9 = \overline{H}\,\overline{K}$. Then, as shown in Table 4, with respect to $\wedge_K$ the set of points Q_h's contains the further point $Q_9 = (x, y, 0)$. In order to prove that the assessment $(x, y, \min\{x, y\})$ is coherent, it is enough to repeat the same reasoning used in the proof of Theorem 3 by only considering the points $Q_1, \ldots, Q_8$ (this amounts to set $\lambda_9 = 0$ in the current system (Σ)). $\square$

Table 4. Constituents C_h's and points Q_h's associated with the prevision assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_L (B|K)\}$.

	C_h	Q_h	
C_1	$AHBK$	$(1, 1, 1)$	Q_1
C_2	$AH\overline{B}K$	$(1, 0, 0)$	Q_2
C_3	$AH\overline{K}$	$(1, y, z)$	Q_3
C_4	$\overline{A}HBK$	$(0, 1, 0)$	Q_4
C_5	$\overline{A}H\overline{B}K$	$(0, 0, 0)$	Q_5
C_6	$\overline{A}H\overline{K}$	$(0, y, 0)$	Q_6
C_7	$\overline{H}BK$	$(x, 1, z)$	Q_7
C_8	$\overline{H}\,\overline{B}K$	$(x, 0, 0)$	Q_8
C_9	$\overline{H}\,\overline{K}$	$(x, y, 0)$	Q_9

3.3 The Bochvar Internal Conjunction

The Bochvar internal conjunction [3] is represented in Table 1 by the symbol $\wedge_B$. As shown in Table 2, based on the betting scheme the conjunction of two conditional events $A|H$ and $B|K$ in our approach coincides with the random quantity

$$(A|H) \wedge_B (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}HBK \vee AH\overline{B}K \vee \overline{A}H\overline{B}K \text{ is true,} \\ z, & \text{if } AH\overline{K} \vee \overline{H}BK \vee \overline{H}\,\overline{K} \vee \overline{A}H\overline{K} \vee \overline{H}\,\overline{B}K \text{ is true,} \end{cases} \tag{11}$$

where $z = P(A|H \wedge_B B|K)$. We observe that $AH\overline{K} \vee \overline{H}BK \vee \overline{H}\,\overline{K} \vee \overline{A}H\overline{K} \vee \overline{H}\,\overline{B}K = \overline{H} \vee \overline{K}$, then

$$(A|H) \wedge_B (B|K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}HBK \vee AH\overline{B}K \vee \overline{A}H\overline{B}K \text{ is true,} \\ z, & \text{if } \overline{H} \vee \overline{K} \text{ is true.} \end{cases}$$

Thus,

$$(A|H) \wedge_B (B|K) = 1 \cdot AHBK + z(\overline{H} \vee \overline{K})\,. \tag{12}$$

Based on (11) and (12), the Bochvar internal conjunction is the following conditional event

$$(A|H) \wedge_B (B|K) = AHBK|HK = AB|HK\,. \tag{13}$$

We have

Theorem 5. *Given any coherent assessment (x, y) on $\{A|H, B|K\}$, with A, H, B, K logically independent, and with $H \neq \emptyset, K \neq \emptyset$, the probability assessment $z = P[(A|H) \wedge_B (B|K)]$ is a coherent extension if and only if $z \in [z', z'']$, where $z' = 0$ and $z'' = 1$.*

Proof. We recall that every $(x, y) \in [0, 1]^2$ is a coherent assessment on $\{A|H, B|K\}$. We will prove that every assessment $(x, y, z) \in [0, 1]^3$ on $\{A|H, B|K, (A|H) \wedge_B (B|K)\}$ is coherent. The constituents C_h's and the points Q_h's associated with $(\mathcal{F}, \mathcal{P})$, where $\mathcal{F} = \{A|H,\, B|K,\, (A|H) \wedge_B (B|K)\}$, $\mathcal{P} = (x, y, z)$, are given in Table 5. We observe that $\mathcal{P}$ belongs to the segment with vertices Q_3, Q_6; indeed $(x, y, z) = x(1, y, z) + (1 - x)(0, y, z)$. $\mathcal{P}$ also belongs to the segment with vertices Q_7, Q_8; indeed $(x, y, z) = y(x, 1, z) + (1 - y)(x, 0, z)$. Then,

$$(x, y, z) = \frac{x}{2}(1, y, z) + \frac{1-x}{2}(0, y, z) + \frac{y}{2}(x, 1, z) + \frac{1-y}{2}(x, 0, z),$$

that is the vector $(\lambda_1, \dots, \lambda_8) = (0, 0, \frac{x}{2}, 0, 0, \frac{1-x}{2}, \frac{y}{2}, \frac{1-y}{2})$ is a solution of system (Σ), with $\sum_{h:C_h \subseteq H} \lambda_h = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = \frac{1}{2} > 0$, $\sum_{h:C_h \subseteq K} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8 = \frac{1}{2} > 0$, and $\sum_{h:C_h \subseteq HK} \lambda_h = \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 = 0$. Therefore $I_0 \subseteq \{3\}$. If $I_0 = \emptyset$, then from Theorem 2 it follows that $(x, y, z) \in [0, 1]^3$ is coherent. If $I_0 = \{3\}$ from coherence of the assessment $P[(A|H) \wedge_B (B|K)] = z \in [0, 1]$ and by Algorithm 1 it follows the coherence of the assessment $(x, y, z) \in [0, 1]^3$. Thus, for every given $(x, y) \in [0, 1]^2$ the lower and upper bounds on the extension $z = P[(A|H) \wedge_B (B|K)]$ are $z' = 0$ and $z'' = 1$, respectively. $\square$

Remark 3. We observe that $(A|HK) \wedge (B|HK) = AB|HK$, so that $P(A|HK)$, $P(B|HK)$, and $P(AB|HK)$ satisfy the Fréchet-Hoeffding bounds, that is

$$\max\{P(A|HK)+P(B|HK)-1, 0\} \leq P(AB|HK) \leq \min\{P(A|HK), P(B|HK)\}.$$

Table 5. Constituents C_h's and points Q_h's associated with the prevision assessment $\mathcal{P} = (x, y, z)$ on $\mathcal{F} = \{A|H, B|K, (A|H) \wedge_B (B|K)\}$.

	C_h	Q_h	
C_1	$AHBK$	$(1, 1, 1)$	Q_1
C_2	$AH\overline{B}K$	$(1, 0, 0)$	Q_2
C_3	$AH\overline{K}$	$(1, y, z)$	Q_3
C_4	$\overline{A}HBK$	$(0, 1, 0)$	Q_4
C_5	$\overline{A}H\overline{B}K$	$(0, 0, 0)$	Q_5
C_6	$\overline{A}H\overline{K}$	$(0, y, z)$	Q_6
C_7	$\overline{H}BK$	$(x, 1, z)$	Q_7
C_8	$\overline{H}\,\overline{B}K$	$(x, 0, z)$	Q_8
C_0	$\overline{H}\,\overline{K}$	(x, y, z)	$Q_0 = \mathcal{P}$

Then, by assuming $P(A|H) = P(A|HK) = x$ and $P(B|K) = P(B|HK) = y$, that is by requiring suitable conditional independence hypotheses, it holds that $z = P(AB|HK)$ is coherent if and only if

$$\max\{P(A|H) + P(B|K) - 1, 0\} \leq P(AB|HK) \leq \min\{P(A|H), P(B|K)\}.$$

A discussion on this aspect related with conditional independence is given in [7], where a general definition of intersection and union between two fuzzy subsets is introduced in the framework of conditional probabilities (see also [6]).

3.4 The Sobocinski Conjunction or Quasi Conjunction

The Sobocinski conjunction or quasi conjunction [1] is represented in Table 1 by the symbol $\wedge_S$. We recall that the link between conditional events and Sobocinski conjunction was studied in [10]. As shown in Table 2, the conjunction of $A|H$ and $B|K$ is the following conditional event

$$(A|H) \wedge_S (B|K) = (AH \vee \overline{H}) \wedge (BK \vee \overline{K})|(H \vee K)\,.$$

Based on the betting scheme the conjunction of two conditional events $A|H$ and $B|K$ in our approach coincides with the random quantity

$$(A|H) \wedge_S (B|K) = \begin{cases} 1, & \text{if } AHBK \vee AH\overline{K} \vee BK\overline{H} \text{ is true,} \\ 0, & \text{if } \overline{A}H \vee \overline{B}K \text{ is true,} \\ z, & \text{if } \overline{H}\,\overline{K} \text{ is true,} \end{cases} \tag{14}$$

From (14), the conjunction $(A|H) \wedge_S (B|K)$ is the following random quantity

$$(A|H) \wedge_S (B|K) = 1 \cdot AHBK + AH\overline{K} + \overline{H}BK + z\overline{H}\,\overline{K}\,. \tag{15}$$

We recall the following result (see, e.g., [18,21])

Theorem 6. *Given any coherent assessment* (x,y) *on* $\{A|H, B|K\}$, *with* A, H, B, K *logically independent, and with* $H \neq \emptyset, K \neq \emptyset$, *the probability assessment* $z = P[(A|H) \wedge_S (B|K)]$ *is a coherent extension if and only if* $z \in [z', z'']$, *where* $z' = \max\{x+y-1, 0\}$ *and* $z'' = \begin{cases} \frac{x+y-2xy}{1-xy}, & (x,y) \neq (1,1), \\ 1, & (x,y) = (1,1). \end{cases}$

We recall that $z'' = S_0^H(x,y)$, where $S_0^H(x,y)$ is the Hamacher t-conorm with parameter $\lambda = 0$. It can be easily verified that

$$S_0^H(x,y) \geq \min\{x,y\}, \ \ \forall (x,y) \in [0,1]^2. \tag{16}$$

4 Conjunction as a Conditional Random Quantity

We recall that the extension $z = P(AB)$ of the assessment (x,y) on $\{A,B\}$, with A, B logically independent, is coherent if and only if z satisfies the Fréchet-Hoeffding bounds, that is $\max\{x+y-1, 0\} \leq z \leq min\{x,y\}$. As we have seen in the previous sections, no one of the given definitions of conjunction between two conditional events preserves both of these lower and upper bounds. A definition of conjunction which satisfies the Fréchet-Hoeffding bounds has been given in recent papers (see, e.g., [22,23]). Based on this definition the conjunction of two conditional events $A|H$ and $B|K$, with $P(A|H) = x$ and $P(B|K) = y$, is the following conditional random quantity

$$(A|H) \wedge (B|K) = \min\{A|H, B|K\}|(H \vee K) = \begin{cases} 1, & \text{if } AHBK \text{ is true}, \\ 0, & \text{if } \overline{A}H \vee \overline{B}K \text{ is true}, \\ x, & \text{if } \overline{H}BK \text{ is true}, \\ y, & \text{if } AH\overline{K} \text{ is true}, \\ z, & \text{if } \overline{H}\,\overline{K} \text{ is true}, \end{cases} \tag{17}$$

where z is the prevision $\mathbb{P}[(A|H) \wedge (B|K)]$ of $(A|H) \wedge (B|K)$. This notion of conjunction satisfies the Fréchet-Hoeffding bounds [22, Theorem 7], that is: given any coherent assessment (x,y) on $\{A|H, B|K\}$, with A, H, B, K logically independent, $H \neq \emptyset, K \neq \emptyset$, the extension $z = \mathbb{P}[(A|H) \wedge (B|K)]$ is coherent if and only if $\max\{x+y-1, 0\} = z' \leq z \leq z'' = \min\{x,y\}$. In case of some logical dependencies among the events A, H, B, K the interval $[z', z'']$ of coherent extensions may be smaller and/or the conjunction may reduce to a conditional event.

For instance, the conjunction $(A|B) \wedge (B|A)$ reduces to the conditional event $AB|(A \vee B)$ [29, Theorem 7]. The disjunction of $A|H$ and $B|K$ is defined as the following conditional random quantity

$$(A|H) \vee (B|K) = \max\{A|H, B|K\}|(H \vee K) = \begin{cases} 1, & \text{if } AH \vee BK \text{ is true}, \\ 0, & \text{if } \overline{A}H\overline{B}K \text{ is true}, \\ x, & \text{if } \overline{H}\,\overline{B}K \text{ is true}, \\ y, & \text{if } \overline{A}H\overline{K} \text{ is true}, \\ w, & \text{if } \overline{H}\,\overline{K} \text{ is true}, \end{cases} \tag{18}$$

where $w = \mathbb{P}[(A|H) \vee (B|K)]$. Given any coherent assessment (x, y) on $\{A|H, B|K\}$, with A, H, B, K logically independent, $H \neq \emptyset, K \neq \emptyset$, the extension $w = \mathbb{P}[(A|H) \vee (B|K)]$ is coherent if and only if $\max\{x, y\} = w'' \leq w \leq w'' = \min\{x + y, 1\}$; that is the classical lower and upper bound for disjunction still hold. Other properties which are satisfied by these notions are the following ones:

- $(A|H) \wedge (B|K) = (B|K) \wedge (A|H)$ and $(A|H) \vee (B|K) = (B|K) \vee (A|H)$;
- $(A|H) \wedge (B|K) \leq A|H$ and $(A|H) \wedge (B|K) \leq B|K$;
- $(A|H) \wedge (A|H) = (A|H) \vee (A|H) = A|H$;
- $A|H \subseteq B|K \iff (A|H) \wedge (B|K) = A|H$ and $(A|H) \vee (B|K) = B|K$;
- $P(A|H) = P(B|K) = 1 \iff \mathbb{P}[(A|H) \wedge (B|K)] = 1$;
- $(A|H) \wedge (B|K) \leq (A|H) \wedge_S (B|K)$, with $(A|H) \wedge (B|K) = (A|H) \wedge_S (B|K)$ when $x = y = 1$.

Definitions (17) and (18) have been generalized to the case of n conditional events in [23] (see also [24]), where it has been shown the validity of the associative properties. Moreover, based on a suitable definition of the negation, it has been shown that De Morgan's Laws hold and that $(A|H) \vee (B|K) = (A|H) + (B|K) - (A|H) \wedge (B|K)$, from which it follows the prevision sum rule: $\mathbb{P}[(A|H) \vee (B|K)] = \mathbb{P}(A|H) + \mathbb{P}(B|K) - \mathbb{P}[(A|H) \wedge (B|K)]$.

Remark 4. Notice that, from Remark 1, Definitions 3 and 17, it follows that $(A|H) \wedge_K (B|K) \leq (A|H) \wedge (B|K)$. Then, it is not surprising that, given (x, y), the lower bound on $(A|H) \wedge_K (B|K)$ (equal to 0) is less than or equal to the lower bound on $(A|H) \wedge (B|K)$, which is $\max\{x + y - 1, 0\}$. A similar comment holds for the Lukasiewicz conjunction, because from (9) it follows that $(A|H) \wedge_L (B|K) \leq (A|H) \wedge (B|K)$. Moreover, as shown in Table 2, in general the inequality $(A|H) \wedge_B (B|K) \leq \min\{A|H, B|K\}$ is not satisfied. Then, it is not surprising that the upper bound on $P[(A|H) \wedge_B (B|K)]$ is greater than or equal to $\min\{x, y\}$; indeed it is equal to 1. Concerning the lower bound, the inequality $(A|H) \wedge_B (B|K) \geq (A|H) \wedge (B|K)$ in general is not satisfied. Then, it is not surprising that the lower bound on $P[(A|H) \wedge_B (B|K)]$ is less than or equal to $\max\{x + y - 1, 0\}$; indeed it is equal to 0. Finally, as $(A|H) \wedge (B|K) \leq (A|H) \wedge_S (B|K)$, it is not surprising that the upper bound on $P[(A|H) \wedge_S (B|K)]$ is greater than or equal to $\min\{x, y\}$; indeed it is $S_0^H(x, y)$ (see formula 16).

5 Conclusions

In this paper we examined four different notions of conjunction among conditional events given in literature such that the result of conjunction is still a conditional event. For each conjunction $(A|H) \wedge (B|K)$, given the conditional probabilities $x = P(A|H)$ and $y = P(B|K)$, we have determined the lower and upper bounds for the conditional probability $z = P[(A|H) \wedge (B|K)]$. We have verified that in all cases such probability bounds do not coincide with the classical Fréchet-Hoeffding bounds. Moreover, we examined a notion of conjunction introduced in some recent papers, where the result of conjunction is (not a conditional event, but) a conditional random quantity. With this notion of conjunction, among other properties, the Fréchet-Hoeffding bounds are satisfied. Further work should concerns the generalization of the results concerning lower and upper bounds for the different definitions of conjunction of n conditional events. A similar study should be made for the notions of disjunction.

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