Probabilistic entailment and iterated conditionals

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Abstract

In this paper we exploit the notions of conjoined and iterated conditionals. These notions are defined, in the setting of coherence, by means of suitable conditional random quantities with values in the interval $[0,1]$. We examine the iterated conditional $(B|K)|(A|H)$, by showing that $A|H$ p-entails $B|K$ if and only if $(B|K)|(A|H) = 1$. Then, we show that a (p-consistent) family $\mathcal{F} = \{E_1|H_1, E_2|H_2\}$ p-entails a conditional event $E_3|H_3$ if and only if $E_3|H_3 = 1$, or $(E_3|H_3)|QC(\mathcal{S}) = 1$ for some nonempty subset $\mathcal{S}$ of $\mathcal{F}$, where $QC(\mathcal{S})$ is the quasi conjunction of the conditional events in $\mathcal{S}$. We also examine the inference rules *And*, *Cut*, *Cautious Monotonicity*, and *Or* of System P, and other well known inference rules (*Modus Ponens*, *Modus Tollens*, and *Bayes*). Furthermore, we show that $QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) = 1$, where $\mathcal{C}(\mathcal{F})$ is the conjunction of the two conditional events in $\mathcal{F}$. We characterize p-entailment by showing that $\mathcal{F}$ p-entails $E_3|H_3$ if and only if $(E_3|H_3)|\mathcal{C}(\mathcal{F}) = 1$. Finally, we examine *Denial of the antecedent*, *Affirmation of the consequent*, and *Transitivity* where the p-entailment of $E_3|H_3$ from $\mathcal{F}$ does not hold, so that $(E_3|H_3)|\mathcal{C}(\mathcal{F}) \neq 1$.

1 Introduction

The new paradigm psychology of reasoning is characterized by using probability theory instead of classical bivalent logic as a normative background theory (see, e.g., Gilio & Over, 2012; Oaksford & Chater, 2007; Over, 2009; Elqayam & Over, 2012; Pfeifer & Douven, 2014; Pfeifer, 2013, to appear; Politzer & Baratgin, 2015). One of the key topics of the new paradigm psychology of reasoning is how people interpret and reason about *conditionals*

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(see, e.g., Douven, 2016; Edgington, 1995; Politzer, Over, & Baratgin, 2010; Evans & Over, 2004; Pfeifer & Kleiter, 2005, 2010; Pfeifer & Tulkki, 2017; Oaksford & Chater, 2003; Over & Cruz, 2018). How people interpret and reason about conditionals was also one of the key topics in the (old) logic-based paradigm psychology of reasoning, which dominated the 20^{th} century experimental psychology of reasoning. While human interpretation of conditionals was labeled as "irrational" or "defective", since the participants' responses deviated from the semantics of the material conditional, rationality was revisited and rehabilitated within the new probabilistic paradigm: specifically, the majority of participants

- treat negated antecedents as irrelevant for evaluating whether a conditional holds, and
- evaluate their degrees of belief in conditionals by respective conditional probabilities (and not by the probability of the material conditional).

These findings speak for the conditional event interpretation (see Section 2), and against the material conditional interpretation, of conditionals.

Among various interpretations of probability, we advocate and use the *coherence-based approach* to probability (see, e.g., Berti, Miranda, & Rigo, 2017; Biazzo & Gilio, 2000; Biazzo, Gilio, Lukasiewicz, & Sanfilippo, 2005; Capotorti, Lad, & Sanfilippo, 2007; Coletti & Scozzafava, 2002; Coletti, Petturiti, & Vantaggi, 2016; Gilio, Pfeifer, & Sanfilippo, 2015, 2016; Gilio & Sanfilippo, 2013c, 2013d, 2014; Pfeifer & Sanfilippo, 2018, 2019; Sanfilippo, 2012; Walley, Pelessoni, & Vicig, 2004), which traces back to Bruno de Finetti (1937/1980, 1970/1974). From a psychological point of view, it is evident that probability serves to measure *degrees of belief* and not some objective quantity in the world: this is in line with de Finetti provocative ontological motto "*Probability does not exist*"(1970/1974, Preface). The probabilistic approach based on coherence is thus characterized by *subjective*, and not by objective, probabilities. Methodologically, the approach based on coherence principle differs in many respects to standard approaches to probabilities. We mention two of them which highlight the psychological plausibility of our approach.

First, contrary to many approaches to probability, the coherence-based approach does not require a complete algebra. For drawing a probabilistic modus ponens inference, for example, an algebra could be constructed from the constituents derived from the involved events in the inference rule. Requiring knowledge about the complete algebra is psychologically implausible, as the reasoning person may focus on only what is considered to be relevant for drawing the inference.

Second, conditional probability is a *primitive* notion and it is not defined by the fraction of the joint and the marginal probabilities: the standard definition of $P(C|A)$ by $\frac{P(A \wedge C)}{P(A)}$ requires to assume that $P(A) > 0$, as a fraction over zero is undefined. Probabilistic approaches which define conditional probabilities in this way can therefore not properly manage zero antecedent probabilities. The subjective probabilistic approach allows for managing zero antecedent probabilities; moreover, zero probabilities are even exploited for reducing the complexity of the probabilistic inference. Another aspect of defining conditional probability *directly* is that the degree of belief in a conditional *If A, then C* can be given in a direct way by the reasoner without presupposing knowledge about $P(A \wedge C)$ and $P(A)$: even though in everyday life it may be impracticable to evaluate the latter two probabilities, people do assess conditionals. For example, if we want to assess our degree of belief in the conditional that *If I take the train at six, I am at home at seven*, we can do that directly, without thinking first about the unconditional probabilities of *I take the train at six and I am at home at seven* and of *I take the train at six*.

In some recent papers of Gilio and Sanfilippo (2013a, 2013b, 2014, 2017) the notions of conjoined and iterated conditionals have been introduced as suitable conditional random quantities. The meaning of conditional random quantity is recalled in Section 2. The conjoined and iterated conditionals properly extend the usual notions of conjunction and conditioning from the case of unconditional events to the case of conditional events (see Section 3).

As an example of a conjoined conditional consider two football matches. For each (valid) match the possible outcomes are: *home win*, *draw*, and *away win*. Then, the conjunction sentence

> *The outcome of the 1st match is home win (if the 1st match is valid) and the outcome of the 2nd is draw (if the 2nd match is valid).*

is a conjoined conditional, because each conjunct is itself a conditional: the first conjunct is the conditional

> *the outcome of the 1st match is home win (if the 1st match is valid)*

and the second conjunct is the conditional

> *the outcome of the 2nd is draw (if the 2nd match is valid).*

As an example of an iterated conditional consider the following conditional sentence (which was presented by Douven, 2016, p. 45):

> (Iter.) *If the mother is angry if the son gets a* B*, then she will be furious if the son gets a* C,

which is an *iterated (or nested) conditional.* It consists of a conditional in its antecedent

> (Ant.) *if the son gets a* B*, then the mother is angry*,

and a conditional in its consequent

> (Cons.) *if the son gets a* C*, then the mother is furious.*

Of course, the degree of belief in (Iter.) cannot be something like a conditional probability, as the famous triviality results by Lewis (1976) have shown. Rather, we conceive iterated conditionals like (Iter.) as conditional random quantities (and not as conditional events) and measure the degree of belief in such objects by previsions $\mathbb{P}$ (not by probabilities P; Gilio & Sanfilippo, 2014; Gilio, Over, Pfeifer, & Sanfilippo, 2017; Sanfilippo, Pfeifer, Over, & Gilio, 2018). We will explain the formal details below. Interestingly, when we considered the uncertainty propagation rule for the generalized probabilistic modus ponens (Sanfilippo, Pfeifer, & Gilio, 2017), where the degree of beliefs are propagated, for instance, from "*The cup broke if dropped*" $(A|H)$, and "if *the cup broke if dropped*, then *the cup was fragile* $(C|(A|H))$" to "*the cup was fragile* (C)", we observed, that the uncertainty propagation rules coincide with those of the non-iterated probabilistic modus ponens (i.e., from $P(A) = x$ and $P(C|A) = y$ infer $xy \leqslant P(C) \leqslant xy + 1 - x$). Likewise, we have shown that the uncertainty propagation rules of the iterated version of Centering coincide with the respective (non-iterated) probability propagation rules (Sanfilippo et al., 2018). Thus, a remarkable aspect of the definitions of nested conditionals in terms of conditional random quantities preserve some well known classical results.

The main result of this paper may be also related to an analogue result derived from the *Deduction Theorem*. This theorem implies that if an argument is logically valid (or if the premises logically entail the conclusion), then the argument can be transformed into a *logically true conditional*, s.t., the premises are combined by conjunction and form the antecedent and the conclusion forms the consequent of the resulting conditional, which is then a tautology. For example, the logically valid modus ponens (where $A \to C$ denotes the material conditional $\bar{A} \vee C$ and $\models$ denotes logical entailment),

$$\{A, A \to C\} \models C\,,$$

can be transformed by the Deduction Theorem into the following conditional, which is a tautology (and *vice versa*), that is:

$$(A \wedge (A \to C)) \to C = \overline{A \wedge (\bar{A} \vee C)} \vee C \;=\; \Omega.$$

Instead of logical entailment, however, we consider in this paper the probabilistic entailment (p-entailment), as introduced by Adams (1975, 1998). Let $\mathcal{C}(\mathcal{F})$ denote the conjunction of the conditional events in a p-consistent family $\mathcal{F}$. We study, in analogy to the Deduction Theorem, whether the claim "a conditional event $E|H$ is p-entailed by a p-consistent family $\mathcal{F}$ of conditional events" is equivalent to the claim "the prevision of the iterated conditional $(E|H)|\mathcal{C}(\mathcal{F})$ is equal to 1". We examine many cases related to this aspect; in particular, we examine some inference rules of System P and other well known inference rules.

We remark that this basic relation, between p-entailment and iterated conditioning, appears in its most elementary form when we consider two not impossible events A and B in the case where $A \subseteq B$, that is where $A \wedge \bar{B} = \emptyset$. In this case $P(A) \leqslant P(B)$ and then A p-entails B, that is $P(A) = 1$ implies $P(B) = 1$, and the unique coherent assessment on $B|A$ is $P(B|A) = 1$. Therefore, by recalling that in the framework of the betting scheme, when we pay $P(B|A) = x$, we receive $B|A = AB + x\bar{A}$, when $A \subseteq B$ it holds that A p-entails B and $B|A = AB + 1 \cdot \bar{A} = A + \bar{A} = 1$. Conversely, if $B|A = 1$, then $P(B|A) = 1$; moreover,

$$P(B) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A}) = P(A) + P(B|\bar{A})P(\bar{A})\,,$$

and when $P(A) = 1$ it follows that $P(B) = 1$, so that A p-entails B.

The outline of the paper is as follows. In Section 2 we recall some preliminary notions and results on coherence, p-entailment and conditional random quantities, as well as conjoined and iterated conditionals. In Section 3 we show that a (p-consistent) conditional event $A|H$ p-entails another (p-consistent) conditional event $B|K$ if and only if $(B|K)|(A|H) = 1$. Moreover, we show that a p-consistent family of two conditional events $\{E_1|H_1, E_2|H_2\}$ p-entails a conditional event $E_3|H_3$ if and only if it holds that $(E_3|H_3)|QC(E_1|H_1, E_2|H_2) = 1$, where $QC(E_1|H_1, E_2|H_2)$ denotes the quasi conjunction of $E_1|H_1, E_2|H_2$. We also characterize p-entailment of $E_3|H_3$ from the family $\{E_1|H_1, E_2|H_2\}$ by the property that $E_3|H_3 = 1$, or $(E_3|H_3)|QC(\mathcal{S}) = 1$ for some nonempty $\mathcal{S} \subseteq \{E_1|H_1, E_2|H_2\}$. In Section 4 we recall a generalized notion of iterated conditioning; then, we examine some inference rules of System P and other well known inference rules. In Section 5 we give two results which relate the notions of conjunction, p-entailment, and iterated conditioning. The first result shows that the iterated conditional having as antecedent and consequent the conjunction and the quasi conjunction of two conditional events, respectively, is equal to 1, i.e., $QC(E_1|H_1, E_2|H_2)|((E_1|H_1) \wedge (E_2|H_2)) = 1$. The second result characterizes the p-entailment of the conditional event $E_3|H_3$ from a p-consistent family $\{E_1|H_1, E_2|H_2\}$ by the property that the iterated conditional $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))$ is equal to 1. Finally, we examine some examples where the p-entailment of the conditional event $E_3|H_3$ from a p-consistent family $\{E_1|H_1, E_2|H_2\}$ does not hold, so that in these cases the iterated conditional $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))$ does not coincide with 1. Long proofs are given in Appendix A.

2 Preliminary notions and results

In our approach events represent uncertain facts described by (non ambiguous) logical propositions. An event A is a two-valued logical entity which is either *true*, or *false*. The indicator of an event A is a two-valued numerical quantity which is 1, or 0, according to whether A is true, or false, respectively. We use the same symbol to refer to an event and its indicator. We denote by Ω the sure event and by $\varnothing$ the impossible one (notice that, when necessary, the symbol $\varnothing$ will denote the empty set). Given two events A and B, we denote by $A \wedge B$, or simply by AB, the intersection, or conjunction, of A and B, as defined in propositional logic; likewise, we denote by $A \vee B$ the union, or disjunction, of A and B. We denote by $\bar{A}$ the negation of A. Of course, the truth values for conjunctions, disjunctions and negations are defined as usual. Given any events A and B, we simply write $A \subseteq B$ to denote that A logically implies B, that is $A\bar{B} = \varnothing$, which means that it is necessary that A and $\bar{B}$ cannot both be true. Given two events E, H, with $H \neq \varnothing$, the conditional event $E|H$ is defined as a three-valued logical entity which is *true*, or *false*, or *void*, according to whether EH is true, or $\bar{E}H$ is true, or $\bar{H}$ is true, respectively. In the betting framework, assessing $P(E|H) = x$ amounts to say that, for every real number s, you are willing to pay an amount $s\,x$ and to receive s, or 0, or $s\,x$, according to whether EH is true, or $\bar{E}H$ is true, or $\bar{H}$ is true (i.e., the bet is called off), respectively. Moreover, for the random gain $G = sH(E-x)$, the possible values are $s(1-x)$, or $-s\,x$, or 0, according to whether EH is true, or $\bar{E}H$ is true, or $\bar{H}$ is true, respectively. More generally speaking, consider a real-valued function $p : \mathcal{K} \to \mathbb{R}$, where $\mathcal{K}$ is an arbitrary (possibly not finite) family of conditional events. Let $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$ be a family of conditional events, where $E_i|H_i \in \mathcal{K}$, $i = 1, \dots, n$, and let $\mathcal{P} = (p_1, \dots, p_n)$ be the vector of values $p_i = P(E_i|H_i)$, where $i = 1, \dots, n$. We denote by $\mathcal{H}_n$ the disjunction $H_1 \vee \cdots \vee H_n$. With the pair $(\mathcal{F}, \mathcal{P})$ we associate the random gain $G = \sum_{i=1}^{n} s_i H_i(E_i - p_i)$, where $s_1, \dots, s_n$ are n arbitrary real numbers. G represents the net gain of n transactions. Let $\mathcal{G}_{\mathcal{H}_n}$ denote the set of possible values of G restricted to $\mathcal{H}_n$, that is, the values of G when at least one conditioning event is true.

Definition 1. *The function P defined on $\mathcal{K}$ is* coherent *if and only if, for every integer n, for every finite subfamily $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$ of $\mathcal{K}$ and for every real numbers $s_1, \dots, s_n$, it holds that:* $\min \mathcal{G}_{\mathcal{H}_n} \leqslant 0 \leqslant \max \mathcal{G}_{\mathcal{H}_n}$.

Intuitively, Definition 1 means in betting terms that a probability assessment is coherent if and only if, in any finite combination of n bets, it cannot happen that the values in $\mathcal{G}_{\mathcal{H}_n}$ are all positive, or all negative (*no Dutch Book*).

Given a conditional event $A|H$ with $P(A|H) = x$, then for (the indicator of) $A|H$ we have $A|H = AH + x\bar{H} \in \{1, 0, x\}$ (Sanfilippo et al., 2018, Appendix A.3). We recall below the notion of logical implication of Goodman and Nguyen (1988) for conditional events (see also Gilio & Sanfilippo, 2013d).

Definition 2. *Given two conditional events $A|H$ and $B|K$ we define that $A|H$* logically implies *$B|K$ (denoted by $A|H \subseteq B|K$) if and only if AH logically implies BK and $\bar{B}K$*

logically implies $\bar{A}H$*; i.e.,* $AH \subseteq BK$ *and* $\bar{B}K \subseteq \bar{A}H$.

A generalization of the Goodman and Nguyen logical implication to conditional random quantities has been given by (Pelessoni & Vicig, 2014).
The notions of p-consistency and p-entailment of Adams (1975) were formulated for conditional events in the setting of coherence by Gilio and Sanfilippo (2013d) (see also Biazzo et al., 2005; Gilio, 2002, 2012; Gilio & Sanfilippo, 2010, 2011, 2013c).

Definition 3. *Let* $\mathcal{F}_n = \{E_i|H_i\ ,\ i = 1, \ldots, n\}$ *be a family of* n *conditional events. Then,* $\mathcal{F}_n$ *is* p-consistent *if and only if the probability assessment* $(p_1, p_2, \ldots, p_n) = (1, 1, \ldots, 1)$ *on* $\mathcal{F}_n$ *is coherent.*

Definition 4. *A p-consistent family* $\mathcal{F}_n = \{E_i|H_i\ ,\ i = 1, \ldots, n\}$ p-entails *a conditional event* $E|H$ *(denoted by* $\mathcal{F}_n \Rightarrow_p E|H$*) if and only if for any coherent probability assessment* $(p_1, \ldots, p_n, z)$ *on* $\mathcal{F}_n \cup \{E|H\}$ *it holds that: if* $p_1 = \cdots = p_n = 1$*, then* $z = 1$.

Of course, when $\mathcal{F}_n$ p-entails $E|H$, there may be coherent assessments $(p_1, \ldots, p_n, z)$ with $z \neq 1$, but in such cases $p_i \neq 1$ for at least one index i. We say that the inference from a p-consistent family $\mathcal{F}_n$ to $E|H$ is *p-valid* if and only if $\mathcal{F}_n$ p-entails $E|H$. We recall the well known notion of quasi conjunction among conditional events:

Definition 5. *Given a family* $\mathcal{F}_n = \{E_i|H_i\ ,\ i = 1, \ldots, n\}$ *of* n *conditional events, the* quasi conjunction *of the conditional events in* $\mathcal{F}_n$ *is defined as*

$$QC(\mathcal{F}_n) = \bigwedge_{i=1}^{n} (\bar{H}_i \vee E_i H_i) | (\bigvee_{i=1}^{n} H_i).$$

Moreover, we recall the following characterization of p-entailment (Gilio & Sanfilippo, 2013c):

Theorem 1. Let a p-consistent family $\mathcal{F}_n = \{E_1|H_1, \ldots, E_n|H_n\}$ and a conditional event $E|H$ be given. The following assertions are equivalent:
1. $\mathcal{F}_n$ p-entails $E|H$;
2. The assessment $\mathcal{P} = (1, \ldots, 1, z)$ on $\mathcal{F} = \mathcal{F}_n \cup \{E|H\}$, where $P(E_i|H_i) = 1$, $i = 1, \ldots, n, P(E|H) = z$, is coherent if and only if $z = 1$;
3. The assessment $\mathcal{P} = (1, \ldots, 1, 0)$ on $\mathcal{F} = \mathcal{F}_n \cup \{E|H\}$, where $P(E_i|H_i) = 1$, $i = 1, \ldots, n, P(E|H) = 0$, is not coherent;
4. Either there exists a nonempty $\mathcal{S} \subseteq \mathcal{F}_n$ such that $QC(\mathcal{S})$ implies $E|H$, or $H \subseteq E$;
5. There exists a nonempty $\mathcal{S} \subseteq \mathcal{F}_n$ such that $QC(\mathcal{S})$ p-entails $E|H$.

We also recall the characterization of the p-entailment for two conditional events (Gilio & Sanfilippo, 2013d, Theorem 7):

Theorem 2. *Given two conditional events* $A|H$, $B|K$*, with* $AH \neq \emptyset$*. It holds that*

$$A|H \Rightarrow_p B|K \Longleftrightarrow A|H \subseteq B|K, \text{ or } K \subseteq B \Longleftrightarrow \Pi \subseteq \{(x, y) \in [0, 1]^2 : x \leqslant y\},$$

where Π *is the set of coherent assessments* (x, y) *on* $\{A|H, B|K\}$.

We denote by X a *random quantity*, that is an uncertain real quantity, which has a well determined but unknown value. We assume that X has a finite set of possible values. Given any event $H \neq \emptyset$, agreeing to the betting metaphor, if you assess that the prevision of

"*X conditional on H*" (or short: "*X given H*"), $\mathbb{P}(X|H)$, is equal to μ, this means that for any given real number s you are willing to pay an amount μs and to receive sX, or μs, according to whether H is true, or false (i.e., when the bet is called off), respectively. In particular, when X is (the indicator of) an event A, then $\mathbb{P}(X|H) = P(A|H)$. Once $\mu = \mathbb{P}(X|H)$ is assessed by the betting scheme, the random quantity $X|H$ can be represented as $X|H = XH + \mu\bar{H}$ (see, e.g., Gilio & Sanfilippo, 2014). The notion of coherence can also be defined for *conditional prevision assessments* on conditional random quantities (for details see, e.g., Biazzo, Gilio, & Sanfilippo, 2012; Gilio & Sanfilippo, 2014; Sanfilippo et al., 2018). In Gilio and Sanfilippo (2014) the notions of conjoined, disjoined, and iterated conditionals have been studied in the framework of conditional random quantities; moreover, the result below (Theorem 4 in that paper) establishes some conditions under which two conditional random quantities $X|H$ and $Y|K$ coincide.

Theorem 3. Given any events $H \neq \varnothing$ and $K \neq \varnothing$, and any random quantities X and Y, let Π be the set of the coherent prevision assessments $\mathbb{P}(X|H) = \mu$ and $\mathbb{P}(Y|K) = \nu$.
(i) Assume that, for every $(\mu, \nu) \in \Pi$, the values of $X|H$ and $Y|K$ always coincide when $H \vee K$ is true; then $\mu = \nu$ for every $(\mu, \nu) \in \Pi$.
(ii) For every $(\mu, \nu) \in \Pi$, the values of $X|H$ and $Y|K$ always coincide when $H \vee K$ is true if and only if $X|H = Y|K$.

We recall the definition of conjunction of two conditional events $A|H$ and $B|K$ (Gilio & Sanfilippo, 2013b, 2013a, 2014). Different approaches to compounded conditionals, not based on coherence, have been developed by other authors (see, e.g., Kaufmann, 2009; McGee, 1989).

Definition 6. Given any pair of conditional events $A|H$ and $B|K$, with $P(A|H) = x$ and $P(B|K) = y$, we define their conjunction as the conditional random quantity

$$(A|H) \wedge (B|K) = (AHBK + x\bar{H}BK + y\bar{K}AH)|(H \vee K) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \bar{A}H \vee \bar{B}K \text{ is true,} \\ x, & \text{if } \bar{H}BK \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ z, & \text{if } \bar{H}\bar{K} \text{ is true,} \end{cases} \tag{1}$$

where z is the prevision of $(A|H) \wedge (B|K)$.

In betting terms, z represents the amount you agree to pay with the proviso that you will receive:

- 1, if both conditional events are true;
- 0, if at least one of the conditional events is false;
- the probability of the conditional event that is void, if one conditional event is void and the other one is true;
- z (i.e., the amount that you payed), if both conditional events are void.

From (1), the conjunction $(A|H) \wedge (B|K)$ can be represented as

$$(A|H) \wedge (B|K) = 1 \cdot AHBK + x \cdot \bar{H}BK + y \cdot AH\bar{K} + z \cdot \bar{H}\bar{K}\,. \tag{2}$$

We observe that if $H = K$, then $\bar{H}BK = AH\bar{K} = \emptyset$, so that $(A|H) \wedge (B|K) = ABH + z\bar{H}$; moreover, $AB|H = ABH + p\bar{H}$, where $p = P(AB|H)$. We notice that $(A|H) \wedge (B|H)$ and $AB|H$ coincide when H is true; then, by Theorem 3, $z = p$; thus,

$$(A|H) \wedge (B|H) = AB|H. \tag{3}$$

We recall that, given any coherent assessment (x, y) on $\{A|H, B|K\}$, with A, H, B, K logically independent, and with $H \neq \emptyset, K \neq \emptyset$, the extension $z = \mathbb{P}[(A|H) \wedge (B|K)]$ is coherent if and only if the following Fréchet-Hoeffding bounds are satisfied (Gilio & Sanfilippo, 2014, Theorem 7):

$$max\{x + y - 1, 0\} = z' \leqslant z \leqslant z'' = min\{x, y\}. \tag{4}$$

Note that the bounds in (4) coincide with the bounds for the conjunction of unconditional probabilities, that is, given any logically independent events A, B and a coherent assessment (x, y) on $\{A, B\}$, the extension $z = P(AB)$ is coherent if and only if $max\{x + y - 1, 0\} \leqslant P(AB) \leqslant min\{x, y\}$. For a study of the lower and upper bounds for other definitions of conjunction see Sanfilippo (2018). The relation between the notions of conjunction and Frank t-norm has been studied in (Gilio & Sanfilippo, 2019a). We now turn to recalling and discussing the notion of iterated conditioning (see, e.g., Gilio & Sanfilippo, 2013a, 2013b, 2014).

Definition 7 (Iterated conditioning)**.** *Given any pair of conditional events $A|H$ and $B|K$, with $AH \neq \emptyset$, the iterated conditional $(B|K)|(A|H)$ is defined as the conditional random quantity*

$$(B|K)|(A|H) = (B|K) \wedge (A|H) + \mu\, \bar{A}|H, \tag{5}$$

where $\mu = \mathbb{P}[(B|K)|(A|H)]$.

Within the betting scheme, to assess $\mathbb{P}[(B|K)|(A|H)] = \mu$ means in particular that you are willing to pay the amount μ, with the proviso that you will receive the quantity $(A|H) \wedge (B|K) + \mu\bar{A}|H$. Of course, this bet requires that you preliminarily evaluate (in a coherent way) the quantities: $x = P(A|H), y = P(B|K), z = \mathbb{P}[(A|H) \wedge (B|K)]$.

Remark 1. *Notice that we assumed that $AH \neq \emptyset$ to give a nontrivial meaning to the notion of the iterated conditional. Indeed, if AH were equal to $\emptyset$, that is $A|H = 0$, then it would be the case that $\bar{A}|H = 1$ and $(B|K)|(A|H) = (B|K)|0 = (B|K) \wedge (A|H) + \mu\bar{A}|H = \mu$ would follow; that is, $(B|K)|(A|H)$ would coincide with the (indeterminate) value μ. Similarly in the case of $B|\emptyset$ (which is of no interest): the trivial iterated conditional $(B|K)|0$ is not considered in our approach.*

We observe that, by linearity of prevision, it holds that

$$\mu = \mathbb{P}((B|K)|(A|H)) = \mathbb{P}((B|K) \wedge (A|H)) + \mu P(\bar{A}|H) = z + \mu(1 - x),$$

from which it follows that $z = \mu x$. Here, when $x > 0$, we obtain $\mu = \frac{z}{x} \in [0, 1]$. Notice that $z + \mu(1 - x)$, i.e. μ, is the value of $(B|K)|(A|H)$ when $\bar{H}\bar{K}$ is true. Then, by observing that

$$\bar{A}H\bar{K} \vee \bar{A}HBK \vee \bar{A}H\bar{B}K \vee \bar{H}\bar{K} = \bar{A}H \vee \bar{H}\bar{K},$$

we obtain

$$(B|K)|(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ x+\mu(1-x), & \text{if } \bar{H}BK \text{ is true,} \\ \mu(1-x), & \text{if } \bar{H}\bar{B}K \text{ is true,} \\ \mu, & \text{if } \bar{A}H\bar{K} \text{ is true,} \\ \mu, & \text{if } \bar{A}HBK \text{ is true,} \\ \mu, & \text{if } \bar{A}H\bar{B}K \text{ is true,} \\ \mu, & \text{if } \bar{H}\bar{K} \text{ is true,} \end{cases} =$$

$$= \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ x+\mu(1-x), & \text{if } \bar{H}BK \text{ is true,} \\ \mu(1-x), & \text{if } \bar{H}\bar{B}K \text{ is true,} \\ \mu, & \text{if } \bar{A}H \vee \bar{H}\bar{K} \text{ is true.} \end{cases}$$

In particular, when $x = 0$, it holds that

$$(B|K)|(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ \mu, & \text{if } \bar{H}BK \text{ is true,} \\ \mu, & \text{if } \bar{H}\bar{B}K \text{ is true,} \\ \mu, & \text{if } \bar{A}H \vee \bar{H}\bar{K} \text{ is true,} \end{cases} = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ \mu, & \text{if } \bar{A}H \vee \bar{H} \text{ is true.} \end{cases}$$

As we can see, in order that the prevision assessment μ on $(B|K)|(A|H)$ be coherent, μ must belong to the convex hull of the values $0, y, 1$; that is, (also when $x = 0$) it must be that $\mu \in [0, 1]$.

3 Quasi conjunction, iteration, and p-entailment of conditionals

In this section we first show that a (p-consistent) conditional $A|H$ p-entails another (p-consistent) conditional $B|K$ if and only if the unique coherent prevision assessment for the corresponding iterated conditional $(B|K)|(A|H)$ is equal to 1, that is, $A|H$ p-entails $B|K$ if and only if $(B|K)|(A|H) = 1$. Then, we show that $\{E_1|H_1, E_2|H_2\}$ p-entails $E_3|H_3$ if and only if $(E_3|H_3)|QC(E_1|H_1, E_2|H_2) = 1$. We first give a preliminary result which relates conjunction and logical implication of Goodman and Nguyen.

Proposition 1. *Given two conditional events $A|H$ and $B|K$, it holds that*

$$A|H \subseteq B|K \Longrightarrow (A|H) \wedge (B|K) = A|H\,. \qquad (6)$$

Proof. We set $P(A|H) = x$, $P(B|K) = y$, and $\mathbb{P}[(A|H) \wedge (B|K)] = z$. As $A|H \subseteq B|K$, it holds that $AH\bar{B}K = AH\bar{K} = \bar{H}\bar{B}K = \emptyset$ and $AHBK = AH$ (Gilio & Sanfilippo, 2013d, Remark 3). Then,

$$(A|H) \wedge (B|K) = AHBK + x\bar{H}BK + y\bar{K}AH + z\bar{H}\bar{K} = AH + x\bar{H}BK + z\bar{H}\bar{K}.$$

Moreover,

$$A|H = AH + x\bar{H} = AH + x\bar{H}BK + x\bar{H}\bar{K}.$$

We notice that $(A|H) \wedge (B|K)$ and $A|H$ coincide when $H \vee K$ is true. Then, $z = x$ follows from Theorem 3. Therefore, $(A|H) \wedge (B|K) = A|H$.

□

Theorem 4. *Given two (p-consistent) conditional events $A|H$ and $B|K$, with $AH \neq \emptyset$, it holds that,*

$$A|H \Rightarrow_p B|K \Longleftrightarrow (B|K)|(A|H) = 1. \tag{7}$$

Proof. ($\Rightarrow$). We distinguish two cases: (i) $A|H \subseteq B|K$; (ii) $K \subseteq B$. Case (i). We remark that if $A|H \subseteq B|K$, then $A|H \leqslant B|K$ and $P(A|H) \leqslant P(B|K)$; moreover, by Proposition 1, $(A|H) \wedge (B|K) = A|H$. Then, by defining $\mathbb{P}((B|K)|(A|H)) = \mu$, $P(A|H) = x$, we obtain

$$\begin{aligned}(B|K)|(A|H) &= (A|H) \wedge (B|K) + \mu\bar{A}|H = A|H + \mu\bar{A}|H = \\ &= \begin{cases} 1, & \text{if } AH \text{ is true,} \\ \mu, & \text{if } \bar{A}H \text{ is true,} \\ x + \mu(1-x), & \text{if } \bar{H} \text{ is true.} \end{cases}\end{aligned}$$

By linearity of prevision, we obtain

$$\mathbb{P}((B|K)|(A|H)) = \mu = P(A|H) + \mu P(\bar{A}|H) = x + \mu(1-x)\,; \tag{8}$$

which implies that

$$(B|K)|(A|H) = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ \mu, & \text{if } \bar{A}H \vee \bar{H} \text{ is true.} \end{cases}$$

In order for μ to be coherent, μ must belong to the convex hull of the set $\{1\}$; i.e. $\mu = 1$. In other words, given two conditional events $A|H$ and $B|K$, with $A|H \subseteq B|K$, it holds that: $\mathbb{P}((B|K)|(A|H)) = 1$. Thus $(B|K)|(A|H) = 1$.
Case (ii). If $K \subseteq B$ it holds that $P(B|K) = y = 1$ and $B|K = 1$. Then, $(A|H) \wedge (B|K) = (A|H)|(H \vee K) = A|H$ (see Gilio & Sanfilippo, 2013a, Remark 4). Moreover, $(B|K)|(A|H) = A|H + \mu\bar{A}|H$ and by linearity of prevision it holds that $\mu = x + \mu(1-x)$. Then,

$$(B|K)|(A|H) = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ \mu, & \text{if } \bar{A}H \text{ is true,} \\ x + \mu(1-x), & \text{if } \bar{H} \text{ is true,} \end{cases} = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ \mu, & \text{if } \bar{A}H \vee \bar{H} \text{ is true.} \end{cases}$$

Then, by coherence, $\mu = 1$ and $(B|K)|(A|H) = 1$.
Thus, p-entailment of $B|K$ from $A|H$ implies $(B|K)|(A|H) = 1$.

($\Leftarrow$). Assume that $(B|K)|(A|H) = 1$, so that the unique coherent assessment for $\mathbb{P}[(B|K)|(A|H)]$ is $\mu = 1$. Then, by observing that $\mathbb{P}[(A|H) \wedge (B|K)] \leqslant P(B|K) = y$, it follows that

$$\mathbb{P}[(A|H) \wedge (B|K)] = \mathbb{P}[(B|K)|(A|H)]P(A|H) = P(A|H) = x \leqslant y.$$

Then, when $x = 1$, it holds that $y = 1$; that is, $A|H$ p-entails $B|K$.

□

Corollary 1. *Let three conditional events* $E_1|H_1$, $E_2|H_2$, *and* $E_3|H_3$ *be given, where* $\{E_1|H_1, E_2|H_2\}$ *is p-consistent. The quasi conjunction* $QC(E_1|H_1, E_2|H_2)$ *p-entails* $E_3|H_3$ *if and only if* $(E_3|H_3)|QC(E_1|H_1, E_2|H_2) = 1$.

Proof. The assertion directly follows by applying Theorem 4, with $A|H = QC(E_1|H_1, E_2|H_2)$ and $B|K = E_3|H_3$. □

In the next result we characterize the p-entailment of $E_3|H_3$ from the family $\{E_1|H_1, E_2|H_2\}$ by the property that $E_3|H_3 = 1$, or $(E_3|H_3)|QC(\mathcal{S}) = 1$ for some nonempty $\mathcal{S} \subseteq \{E_1|H_1, E_2|H_2\}$.

Theorem 5. *Let three conditional events* $E_1|H_1$, $E_2|H_2$, *and* $E_3|H_3$ *be given, where* $\{E_1|H_1, E_2|H_2\}$ *is p-consistent. Then, the family* $\{E_1|H_1, E_2|H_2\}$ *p-entails* $E_3|H_3$ *if and only if at least one of the following conditions is satisfied: (i)* $E_3|H_3 = 1$*; (ii)* $(E_3|H_3)|(E_1|H_1) = 1$*; (iii)* $(E_3|H_3)|(E_2|H_2) = 1$*; (iv)* $(E_3|H_3)|QC(E_1|H_1, E_2|H_2) = 1$.

Proof. ($\Rightarrow$). By Theorem 1, as $\{E_1|H_1, E_2|H_2\}$ p-entails $E_3|H_3$, it follows that $QC(\mathcal{S}) \subseteq E_3|H_3$ for some $\varnothing \neq \mathcal{S} \subseteq \{E_1|H_1, E_2|H_2\}$, or $H_3 \subseteq E_3$. If $H_3 \subseteq E_3$, then $P(E_3|H_3) = 1$ and $E_3|H_3 = 1$. If $\mathcal{S} = \{E_i|H_i\}$, for $i = 1$ or $i = 2$, by Theorem 4 it holds that $(E_3|H_3)|(E_i|H_i) = 1$. If $\mathcal{S} = \{E_1|H_1, E_2|H_2\}$, then by Corollary 1 it holds that $(E_3|H_3)|QC(E_1|H_1, E_2|H_2) = 1$.
($\Leftarrow$). If $E_3|H_3 = 1$ then the unique coherent assessment on $E_3|H_3$ is $P(E_3|H_3) = 1$. This means that $H_3 \subseteq E_3$ and then $\{E_1|H_1, E_2|H_2\}$ p-entails $E_3|H_3$.
If $(E_3|H_3)|(E_i|H_i) = 1$, for $i = 1$ or $i = 2$, then by Theorem 4 it holds that $E_i|H_i$ p-entails $E_3|H_3$ and hence, by Theorem 1, $\{E_1|H_1, E_2|H_2\}$ p-entails $E_3|H_3$.
Finally, if $(E_3|H_3)|QC(E_1|H_1, E_2|H_2) = 1$, then by Corollary 1 it holds that $QC(E_1|H_1, E_2|H_2)$ p-entails $E_3|H_3$ and hence, by Theorem 1, $\{E_1|H_1, E_2|H_2\}$ p-entails $E_3|H_3$.

□

4 Iterated conditionals and some inference rules

In this section we examine some inference rules with $\{E_1|H_1, E_2|H_2\}$ as the premise set, and $E_3|H_3$ as the conclusion, by showing that, if $\{E_1|H_1, E_2|H_2\} \Rightarrow_p E_3|H_3$, then $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1$, which means that the conclusion *given* the conjunction of the premises is constant and equal to 1. Note that the meaning of the term "*given*" will be clarified in Definition 9. We recall below the notion of conjunction of three conditional events (Gilio & Sanfilippo, 2017).

Definition 8. *Given a family of three conditional events* $\mathcal{F} = \{E_1|H_1, E_2|H_2, E_3|H_3\}$*, we set* $P(E_i|H_i) = x_i$, $i = 1, 2, 3$, $\mathbb{P}[(E_i|H_i) \wedge (E_j|H_j)] = x_{ij} = x_{ji}$, $i \neq j$. *The conjunction*

$\mathcal{C}(\mathcal{F}) = (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)$ *is defined as the conditional random quantity*

$$\mathcal{C}(\mathcal{F}) = (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3) = \begin{cases} 1, & \text{if } E_1H_1E_2H_2E_3H_3 \text{ is true,} \\ 0, & \text{if } \bar{E}_1H_1 \vee \bar{E}_2H_2 \vee \bar{E}_3H_3 \text{ is true,} \\ x_1, & \text{if } \bar{H}_1E_2H_2E_3H_3 \text{ is true,} \\ x_2, & \text{if } E_1H_1\bar{H}_2E_3H_3 \text{ is true,} \\ x_3, & \text{if } E_1H_1E_2H_2\bar{H}_3 \text{ is true,} \\ x_{12}, & \text{if } \bar{H}_1\bar{H}_2E_3H_3 \text{ is true,} \\ x_{13}, & \text{if } \bar{H}_1E_2H_2\bar{H}_3 \text{ is true,} \\ x_{23}, & \text{if } E_1H_1\bar{H}_2\bar{H}_3 \text{ is true,} \\ x_{123}, & \text{if } \bar{H}_1\bar{H}_2\bar{H}_3 \text{ is true,} \end{cases} \tag{9}$$

where $x_{123} = \mathbb{P}[\mathcal{C}(\mathcal{F})]$ *and the conditioning event is* $H_1 \vee H_2 \vee H_3$.

Within the betting scheme, to assess $\mathbb{P}[\mathcal{C}(\mathcal{F})] = x_{123}$ means in particular that you are willing to pay the amount x_{123}, with the proviso that you will receive the quantity $\mathcal{C}(\mathcal{F})$. Of course, this bet requires that you preliminarily evaluate (in a coherent way) the quantities x_1, x_2, x_{12}, x_{13}, and x_{23}.

We define below the object $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))$, which has been introduced in general in (Gilio & Sanfilippo, 2019b, Definition 14).

Definition 9. *Let three conditional events $E_1|H_1$, $E_2|H_2$, and $E_3|H_3$ be given, with $(E_1|H_1) \wedge (E_2|H_2) \neq 0$. We denote by $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))$ the conditional random quantity*

$$(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3) + \mu(1 - (E_1|H_1) \wedge (E_2|H_2)),$$

where $\mu = \mathbb{P}[(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))]$.

Remark 2. *We observe that, defining $\mathbb{P}[(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)] = t$ and $\mathbb{P}[(E_1|H_1) \wedge (E_2|H_2)] = z$, by the linearity of prevision it holds that $\mu = t + \mu(1 - z)$; then, $t = \mu z$, that is*

$$\begin{aligned} &\mathbb{P}[(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)] = \\ &= \mathbb{P}[(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))]\mathbb{P}[(E_1|H_1) \wedge (E_2|H_2)]. \end{aligned}$$

4.1 Valid conditional syllogisms

We examine the p-valid conditional syllogisms Modus Ponens and Modus Tollens by instantiations of $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1$, where $\{E_1|H_1, E_2|H_2\}$ is the premise set and $E_3|H_3$ is the conclusion. We note that an unconditional event A, which denotes the unconditional premise of Modus Ponens, coincides with the conditional event $A|\Omega$. Likewise, the unconditional premise of Modus Tollens $\bar{C}$ coincides with $\bar{C}|\Omega$.

Modus Ponens: $\{C|A, A\} \Rightarrow_p C$. It holds that $(C|A) \wedge A = AC = QC(AC)$; then, by Theorem 4, as $AC \subseteq C$ it follows that

$$C|((C|A) \wedge A) = C|(QC((C|A), A) = C|AC = 1\,,$$

this means that the conclusion of the Modus Ponens given the conjunction of the premises is constant and coincides with 1. This can be seen as an analogy to the fact that the modus ponens is logically valid in logic and that the probabilistic modus ponens is p-valid.

Modus Tollens: $\{C|A, \bar{C}\} \Rightarrow_p \bar{A}$. It holds that $(C|A) \wedge \bar{C} = x\bar{A}\bar{C}$, where $x = P(C|A)$, while $QC(C|A, \bar{C}) = \bar{A}\bar{C}$; then, assuming $x > 0$, we obtain

$$\bar{A}|((C|A) \wedge \bar{C}) = \bar{A} \wedge (C|A) \wedge \bar{C} + \mu(1 - (C|A) \wedge \bar{C})) = \begin{cases} \mu, & \text{if } A \vee C \text{ is true,} \\ x + \mu(1-x), & \text{if } \bar{A}\bar{C} \text{ is true.} \end{cases}$$

By coherence it must be the case that $\mu = x + \mu(1 - x)$, i.e., $x = \mu x$, which implies $\mu = x + \mu(1 - x) = 1$; therefore,

$$\bar{A}|((C|A) \wedge \bar{C}) = 1\,.$$

This can be seen as an analogy to the fact that the modus tollens is logically valid in logic and that the probabilistic modus tollens is p-valid. Notice that, if $x = 0$, then $((C|A) \wedge \bar{C}) = 0$ and $\bar{A}|((C|A) \wedge \bar{C}) = \bar{A}|0 = \mu$, which is indeterminate (see Remark 1).

4.2 Bayes rule

We now show the p-entailment of $H|EA$ from $\{E|AH, H|A\}$, that is the p-validity of the Bayes rule. Then, we verify that $(H|EA)|((E|HA) \wedge (H|A)) = 1$.

Bayes. We note that $(E|AH) \wedge (H|A) = EH|A = QC(E|AH, H|A)$; then, as $EH|A \subseteq H|EA$, by Theorem 1 it holds that $\{E|AH, H|A\} \Rightarrow_p H|EA$. Moreover, by Theorem 4, it follows that

$$(H|EA)|((E|HA) \wedge (H|A)) = (H|EA)|QC(E|HA, H|A) = (H|EA)|(EH|A) = 1\,.$$

In particular, if $A = \Omega$, we obtain $(H|E)|((E|H) \wedge H) = (H|E)|(EH) = 1$.

4.3 And, Cut, Cautious Monotonicity, and Or of System P

In this section we consider the following inference rules of System P (Kraus, Lehmann, & Magidor, 1990): And, Cut, Cautious Monotonicity (short: CM), and Or. System P is a basic nonmonotonic reasoning which allows for retracting conclusions in the light of new premises. The probabilistic versions of the rules of System P are p-valid (Adams, 1975; Biazzo, Gilio, Lukasiewicz, & Sanfilippo, 2002; Gilio, 2002). Experimental evidence supports the psychological plausibility of System P (see, e.g. Da Silva Neves, Bonnefon, & Raufaste, 2002; Pfeifer & Kleiter, 2003, 2005; Schurz, 2005).

And rule: $\{B|A, C|A\} \Rightarrow_p BC|A$. By formula (3), it holds that $(B|A) \wedge (C|A) = BC|A = QC(B|A, C|A)$; then, by Theorem 4, as $BC|A \subseteq BC|A$ it follows that

$$(BC|A)|((C|A) \wedge (B|A)) = (BC|A)|QC(B|A, C|A) = (BC|A)|(BC|A) = 1\,.$$

Cut rule: $\{C|AB, B|A\} \Rightarrow_p C|A$. We note that $(C|AB) \wedge (B|A) = BC|A = QC(C|AB, B|A)$; then, by Theorem 4, as $BC|A \subseteq C|A$ it follows that

$$(C|A)|((C|AB) \wedge (B|A)) = (C|A)|QC(C|AB, B|A) = (C|A)|(BC|A) = 1\,.$$

CM rule: $\{C|A, B|A\} \Rightarrow_p C|AB$. By formula (3), it holds that $(C|A) \wedge (B|A) = BC|A = QC(C|A, B|A)$; then, by Theorem 4, as $BC|A \subseteq C|AB$ it follows that

$$(C|AB)|((C|A) \wedge (B|A)) = (C|AB)|QC(C|A, B|A) = (C|AB)|(BC|A) = 1\,.$$

Or rule: $\{C|A, C|B\} \Rightarrow_p C|(A \vee B)$. The next result shows that the conclusion of the Or rule, $C|(A \vee B)$, given the conjunction of the premises, $(C|A) \wedge (C|B)$, coincides with 1.

Theorem 6. *Given a p-consistent family* $\{C|A, C|B\}$ *it holds that*

$$(C|(A \vee B)|((C|A) \wedge (C|B)) = 1.$$

Proof. See Appendix A. □

Remark 3. *We observe that*

$$QC(C|A, C|B) = ((\bar{A} \vee C) \wedge (\bar{B} \vee C))|(A \vee B) = C|(A \vee B).$$

Then, the statement of Theorem 6 amounts to say that the iterated conditional $QC(C|A, C|B)|((C|A) \wedge (C|B))$ *is equal to 1. This aspect will be analyzed in general in the next section.*

5 Conjunction, iteration, and p-entailment of conditionals

In this section we give two results which relate p-entailment, conjunction, and iterated conditioning. Then, we give some examples of non p-valid argument forms. In the next theorem, by defining $\mathcal{F} = \{E_1|H_1, E_2|H_2\}$, $QC(\mathcal{F}) = QC(E_1|H_1, E_2|H_2)$ and $\mathcal{C}(\mathcal{F}) = (E_1|H_1) \wedge (E_2|H_2)$, we show that, under p-consistency of $\mathcal{F}$, the iterated conditional $QC(\mathcal{F})|(\mathcal{C}(\mathcal{F}))$ is equal to 1.

Theorem 7. *Given a p-consistent family* $\mathcal{F} = \{E_1|H_1, E_2|H_2\}$, *it holds that* $QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) = 1$.

Proof. See Appendix A. □

The next theorem shows that the p-entailment of a conditional event $E_3|H_3$ from a p-consistent family $\{E_1|H_1, E_2|H_2\}$ is equivalent to the iterated conditional $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))$ being constant and equal to 1, i.e., the set of possible values of $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))$ is the singleton $\{1\}$.

Theorem 8. *Let three conditional events* $E_1|H_1$, $E_2|H_2$, *and* $E_3|H_3$ *be given, where* $\{E_1|H_1, E_2|H_2\}$ *is p-consistent. Then,* $\{E_1|H_1, E_2|H_2\}$ *p-entails* $E_3|H_3$ *if and only if* $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1$.

Proof. See Appendix A. □

Remark 4. *We recall that* $\{E_1|H_1, E_2|H_2\}$ *p-entails* $QC(E_1|H_1, E_2|H_2)$ *(QAND rule, see, e.g., Gilio & Sanfilippo, 2011, 2013c). Then, Theorem 7 follows by applying Theorem 8 with* $E_3|H_3 = QC(E_1|H_1, E_2|H_2)$. *Similar comments can be made for the inference rules examined in Section 4.*

In the examples below we show that if $\{E_1|H_1, E_2|H_2\}$ does not p-entail $E_3|H_3$, the iterated conditional $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))$ does not coincide with 1. This means that the set of possible values of the iterated conditional $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))$ is not the singleton $\{1\}$.

Example 1 (Denial of the antecedent)**.** *We consider the rule where the premise set is* $\{\bar{A}, C|A\}$ *and the conclusion is* $\bar{C}$. *As is well known, Denial of the antecedent is neither logically valid in logic nor p-valid in probability logic. Indeed, by defining* $P(\bar{A}) = x, P(C|A) = y, P(\bar{C}) = z$, *it holds that*

$$P(\bar{C}) = z = 1 - P(C) = 1 - [P(C|A)P(A) + P(C|\bar{A})P(\bar{A})] = 1 - y(1-x) - P(C|\bar{A})x;$$

Then, when $x = y = 1$, *we obtain* $z = 1 - P(C|\bar{A}) \in [0,1]$*; thus,* $\{\bar{A}, C|A\}$ *does not p-entail* $\bar{C}$. *Then, by Theorem 8, the iterated conditional* $\bar{C}|(\bar{A} \wedge (C|A))$ *does not coincide with 1. Indeed, by defining* $\mathbb{P}[\bar{C}|(\bar{A} \wedge (C|A))] = \mu$, *it holds that*

$$\bar{C}|(\bar{A} \wedge (C|A)) = \bar{C} \wedge \bar{A} \wedge (C|A) + \mu(1 - \bar{A} \wedge (C|A)) = \begin{cases} \mu, & \text{if } AC \text{ is true,} \\ \mu, & \text{if } A\bar{C} \text{ is true,} \\ \mu(1-y), & \text{if } \bar{A}C \text{ is true,} \\ y + \mu(1-y), & \text{if } \bar{A}\bar{C} \text{ is true.} \end{cases}$$

If $y = 1$, *we obtain*

$$\bar{C}|(\bar{A} \wedge (C|A)) = \begin{cases} \mu, & \text{if } AC \text{ is true,} \\ \mu, & \text{if } A\bar{C} \text{ is true,} \\ 1, & \text{if } \bar{A}C \text{ is true,} \\ 0, & \text{if } \bar{A}\bar{C} \text{ is true,} \end{cases}$$

with μ *being coherent, for every* $\mu \in [0,1]$. *Therefore,* $\bar{C}|(\bar{A} \wedge (C|A)) \neq 1$.

Example 2 (Affirmation of the consequent)**.** *We consider the rule where the premise set is* $\{C, C|A\}$ *and the conclusion is* A. *Affirmation of the consequent is neither logically valid in logic nor p-valid in probability logic. Indeed, by defining* $P(C) = x, P(C|A) = y, P(A) = z$, *and* $P(C|\bar{A}) = t$, *it holds that*

$$P(C) = x = P(C|A)P(A) + P(C|\bar{A})P(\bar{A}) = yz + t(1-z).$$

Then, when $x = y = 1$, *we obtain* $1 = z + t - zt$, *that is* $z(1-t) = (1-t)$. *Therefore, when* $t < 1$, *it follows that* $z = 1$. *In other words, by adding the premise* $P(C|\bar{A}) < 1$ *(i.e. what we introduced as a* negated default *in Gilio et al., 2016), it holds that*

$$P(C) = 1, P(C|A) = 1, P(C|\bar{A}) < 1 \Rightarrow P(A) = 1.$$

But in general (where no assumptions are made about $P(C|\bar{A})$*),* $z \in [0,1]$*; thus p-entailment of* A *from* $\{C, C|A\}$ *does not hold. Then, by Theorem 8, the iterated conditional* $A|(C \wedge (C|A))$ *does not coincide with 1. Indeed, by defining* $\mathbb{P}[A|(C \wedge (C|A))] = \mu$, *it holds that*

$$A|(C \wedge (C|A)) = A \wedge C \wedge (C|A) + \mu(1 - C \wedge (C|A)) = \begin{cases} 1, & \text{if } AC \text{ is true,} \\ \mu(1-y), & \text{if } \bar{A}C \text{ is true,} \\ \mu, & \text{if } \bar{C} \text{ is true.} \end{cases}$$

If $y = 1$, *we obtain*

$$A|(C \wedge (C|A)) = \begin{cases} 1, & \text{if } AC \text{ is true,} \\ 0, & \text{if } \bar{A}C \text{ is true,} \\ \mu, & \text{if } \bar{C} \text{ is true.} \end{cases}$$

with μ *being coherent, for every* $\mu \in [0,1]$. *Therefore,* $A|(C \wedge (C|A)) \neq 1$.

Example 3 (Transitivity)**.** *We consider the rule where* $\{C|B, B|A\}$ *is the premise set and* $C|A$ *is the conclusion. Transitivity is not p-valid. Indeed, it can be verified that the assessment* $(1, 1)$ *on* $\{C|B, B|A\}$ *is coherent, that is* $\{C|B, B|A\}$ *is p-consistent; moreover, the assessment* $(1, 1, 0)$ *on* $\{C|B, B|A, C|A\}$ *is coherent and then, by Theorem 1,* $\{C|B, B|A, C|A\}$ *does not p-entail* $C|A$*. Thus, by Theorem 8, the iterated conditional* $(C|A)|((C|B) \wedge (B|A))$ *does not coincide with 1. We also point out that by adding the negated default* $P(\bar{A}|(A \vee B)) < 1$ *it holds that (Gilio et al., 2016, Theorem 5)*

$$P(C|B) = 1, P(B|A) = 1, P(\bar{A}|(A \vee B)) < 1 \Rightarrow P(C|A) = 1,$$

which is a probabilistic version of weak transitivity.

6 Concluding remarks

The results of this paper are based on the notions of conjoined conditionals and iterated conditionals. These objects, introduced in recent papers by Gilio and Sanfilippo, are defined in the setting of coherence by means of suitable conditional random quantities with values in the interval $[0, 1]$. By exploiting the logical implication of Goodman and Nguyen we have shown that, given two conditional events $A|H$ and $B|K$ with $AH \neq \emptyset$ and $BK \neq \emptyset$, $A|H$ p-entails $B|K$ if and only if $(B|K)|(A|H) = 1$. An analogy to this result can be derived from the Deduction Theorem in logic: if the material conditional $K \to B$ follows logically from $H \to A$, then the (nested) material conditional $(H \to A) \to (K \to B)$ is a tautology and *vice versa*. Moreover, we have shown that a p-consistent family $\mathcal{F} = \{E_1|H_1, E_2|H_2\}$ p-entails a conditional event $E_3|H_3$ if and only if $E_3|H_3 = 1$, or $(E_3|H_3)|QC(\mathcal{S}) = 1$ for some nonempty subset $\mathcal{S}$ of $\mathcal{F}$. We have also examined the inference rules And, Cut, Cautious Monotonicity, and Or of System P, and the inference rules Modus Ponens, Modus Tollens, and Bayes. We have shown that the iterated conditional $QC(\mathcal{F})|\mathcal{C}(\mathcal{F})$ is equal to 1 for every p-consistent family $\mathcal{F} = \{E_1|H_1, E_2|H_2\}$. Then, we have characterized the p-entailment of $E_3|H_3$ from a p-consistent family $\mathcal{F}$ by showing that it is equivalent to the condition $(E_3|H_3)|\mathcal{C}(\mathcal{F}) = 1$. Moreover, we have examined Denial of the Antecedent, Affirmation of the Consequent and Transitivity. We have shown that for these argument forms the p-entailment of the conclusion $E_3|H_3$ from a p-consistent premise set $\{E_1|H_1, E_2|H_2\}$ does not hold, so that $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) \neq 1$. In particular, concerning the Affirmation of the Consequent and Transitivity, we have also shown that (a kind of conditional) p-entailment holds if we add a suitable negated default to the premise set. Psychologically, this could serve as a new explanation why some people interpret Affirmation of the Consequent and Transitivity as valid argument forms. Indeed, these argument forms play an important role in abductive reasoning in philosophy of science (e.g., where conclusions about possible causes/diseases are derived from effects/symptoms). Future work is needed to explore such applications of the presented theory and to explore further formal desiderata also related to the Deduction Theorem.

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A Appendix

In this appendix we give the proofs of Theorems 6, 7, and 8.

Proof. of Theorem 6.
By Definition 9, we obtain

$$(C|(A \vee B))|((C|A) \wedge (C|B)) = (C|(A \vee B)) \wedge (C|A) \wedge (C|B) + \mu[1 - (C|A) \wedge (C|B)]\,,$$

where $\mu = \mathbb{P}[(C|(A \vee B))|(C|A) \wedge (C|B)]$. We set $P(C|A) = x$, $P(C|B) = y$, and $\mathbb{P}((C|A) \wedge (C|B)) = z$; then,

$$(C|A) \wedge (C|B) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } (A \vee B)\bar{C} \text{ is true,} \\ x, & \text{if } \bar{A}BC \text{ is true,} \\ y, & \text{if } A\bar{B}C \text{ is true,} \\ z, & \text{if } \bar{A}\bar{B} \text{ is true.} \end{cases}$$

Moreover, by defining $\mathbb{P}[(C|(A \vee B)) \wedge (C|A) \wedge (C|B)] = t$, we obtain

$$(C|(A \vee B)) \wedge (C|A) \wedge (C|B) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } (A \vee B)\bar{C} \text{ is true,} \\ x, & \text{if } \bar{A}BC \text{ is true,} \\ y, & \text{if } A\bar{B}C \text{ is true,} \\ t, & \text{if } \bar{A}\bar{B} \text{ is true.} \end{cases}$$

As we can see, $(C|(A \vee B)) \wedge (C|A) \wedge (C|B)$ and $(C|A) \wedge (C|B)$ coincide when $A \vee B$ is true; then, by Theorem 3 it holds that $t = z$, so that

$$(C|(A \vee B)) \wedge (C|A) \wedge (C|B) = (C|A) \wedge (C|B).$$

Then,

$$(C|(A \vee B))|((C|A) \wedge (C|B)) = (C|A) \wedge (C|B) + \mu[1 - (C|A) \wedge (C|B)]\,, \tag{10}$$

and by the linearity of prevision we obtain $\mu = z + \mu(1 - z)$, so that $z = \mu z$. Moreover, by (10) we obtain

$$(C|(A \vee B))|((C|A) \wedge (C|B)) \quad = \quad \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ x + \mu(1-x), & \text{if } \bar{A}BC \text{ is true,} \\ y + \mu(1-y), & \text{if } A\bar{B}C \text{ is true,} \\ \mu, & \text{if } AB\bar{C} \text{ is true,} \\ \mu, & \text{if } \bar{A}B\bar{C} \text{ is true,} \\ \mu, & \text{if } A\bar{B}\bar{C} \text{ is true,} \\ \mu, & \text{if } \bar{A}\bar{B} \text{ is true,} \end{cases}$$

which reduces to

$$(C|(A \vee B))|((C|A) \wedge (C|B)) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ x + \mu(1-x), & \text{if } \bar{A}BC \text{ is true,} \\ y + \mu(1-y), & \text{if } A\bar{B}C \text{ is true,} \\ \mu, & \text{if } \bar{A}\bar{B}C \vee \bar{C} \text{ is true.} \end{cases}$$

In order to prove that $(C|(A \vee B))|((C|A) \wedge (C|B)) = 1$, we distinguish the following cases: (a) $z > 0$; (b) $z = x = y = 0$; (c) $z = 0, x > 0, y > 0$; (d) $z = y = 0, x > 0$; (e) $z = x = 0, y > 0$.
Case (a). By recalling that $z = \mu z$, as $z > 0$ it follows that $\mu = 1$; then, $y + \mu(1-y) = x + \mu(1-x) = 1$, so that

$$(C|(A \vee B))|((C|A) \wedge (C|B)) = \begin{cases} 1, & \text{if } ABC \vee A\bar{B}C \vee \bar{A}BC \text{ is true}, \\ \mu, & \text{if } \bar{A}\bar{B}C \vee \bar{C} \text{ is true}. \end{cases}$$

Then, by coherence, $\mu = 1$ and $(C|(A \vee B))|((C|A) \wedge (C|B)) = 1$.
Case (b). As $x = y = 0$, it holds that $x + \mu(1-x) = y + \mu(1-y) = \mu$; then

$$(C|(A \vee B))|((C|A) \wedge (C|B)) = \begin{cases} 1, & \text{if } ABC \text{ is true}, \\ \mu, & \text{if } \overline{ABC} \text{ is true}. \end{cases},$$

and, by coherence, $\mu = 1$; thus, $(C|(A \vee B))|((C|A) \wedge (C|B)) = 1$.
Case (c). By coherence, μ is a linear convex combination of the values 1, $y + \mu(1-y)$, and $x + \mu(1-x)$, that is,

$$\mu = \lambda_1 + \lambda_2(y + \mu(1-y)) + \lambda_3(x + \mu(1-x)), \tag{11}$$

with $\lambda_h \geqslant 0, h = 1,2,3$, and $\lambda_1 + \lambda_2 + \lambda_3 = 1$. The equation (11) can be written as

$$\mu(\lambda_1 + \lambda_2 y + \lambda_3 x) = \lambda_1 + \lambda_2 y + \lambda_3 x,$$

where $\lambda_1 + \lambda_2 y + \lambda_3 x > 0$; then $\mu = y + \mu(1-y) = x + \mu(1-x) = 1$ and $(C|(A \vee B))|((C|A) \wedge (C|B)) = 1$.
Case (d). As $y = 0$ it holds that $y + \mu(1-y) = \mu$; then,

$$(C|(A \vee B))|((C|A) \wedge (C|B)) = \begin{cases} 1, & \text{if } ABC \text{ is true}, \\ x + \mu(1-x), & \text{if } \bar{A}BC \text{ is true}, \\ \mu, & \text{if } \overline{BC} \text{ is true}. \end{cases}$$

By coherence, μ is a linear convex combination of the values $1, x + \mu(1-x)$, that is

$$\mu = \lambda_1 + \lambda_2(x + \mu(1-x)), \ \lambda_1 \geqslant 0, \ \lambda_2 \geqslant 0, \ \lambda_1 + \lambda_2 = 1. \tag{12}$$

The equation (12) can be written as $\mu(\lambda_1 + \lambda_2 x) = \lambda_1 + \lambda_2 x$, where $\lambda_1 + \lambda_2 x > 0$; then, $\mu = x + \mu(1-x) = 1$ and $(C|(A \vee B))|((C|A) \wedge (C|B)) = 1$.
Case (e). Since $x = 0$, it holds that $x + \mu(1-x) = \mu$; then,

$$(C|(A \vee B))|((C|A) \wedge (C|B)) = \begin{cases} 1, & \text{if } ABC \text{ is true}, \\ y + \mu(1-y), & \text{if } A\bar{B}C \text{ is true}, \\ \mu, & \text{if } \overline{AC} \text{ is true}. \end{cases}$$

By coherence, μ is a linear convex combination of the values $1, y + \mu(1-y)$, that is

$$\mu = \lambda_1 + \lambda_2(y + \mu(1-y)), \ \lambda_1 \geqslant 0, \ \lambda_2 \geqslant 0, \ \lambda_1 + \lambda_2 = 1. \tag{13}$$

The equation (13) can be written as $\mu(\lambda_1 + \lambda_2 y) = \lambda_1 + \lambda_2 y$, where $\lambda_1 + \lambda_2 y > 0$; then, $\mu = y + \mu(1-y) = 1$ and $(C|(A \vee B))|((C|A) \wedge (C|B)) = 1$.

□

C_h	$QC(\mathcal{F})$	$\mathcal{C}(\mathcal{F})$	$\mathcal{C}(\mathcal{F}) \wedge QC(\mathcal{F})$
$E_1H_1E_2H_2$	1	1	1
$E_1H_1\bar{E}_2H_2$	0	0	0
$E_1H_1\bar{H}_2$	1	x_2	x_2
$\bar{E}_1H_1E_2H_2$	0	0	0
$\bar{E}_1H_1\bar{E}_2H_2$	0	0	0
$\bar{E}_1H_1\bar{H}_2$	0	0	0
$\bar{H}_1E_2H_2$	1	x_1	x_1
$\bar{H}_1\bar{E}_2H_2$	0	0	0
$\bar{H}_1\bar{H}_2$	ν_{12}	x_{12}	η

Table 1: Possible values of the random vector $(QC(\mathcal{F}), \mathcal{C}(\mathcal{F}), \mathcal{C}(\mathcal{F}) \wedge QC(\mathcal{F}))$.

Proof. of Theorem 7.
We recall that $\mathcal{F} = \{E_1|H_1, E_2|H_2\}$. We set $P(E_1|H_1) = x_1$, $P(E_2|H_2) = x_2$, $P(QC(\mathcal{F})) = \nu_{12}$, $\mathbb{P}(\mathcal{C}(\mathcal{F})) = x_{12}$, $\mathbb{P}[\mathcal{C}(\mathcal{F}) \wedge QC(\mathcal{F})] = \eta$, and $\mathbb{P}[QC(\mathcal{F})|\mathcal{C}(\mathcal{F})] = \mu$. Then,

$$QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) = \mathcal{C}(\mathcal{F}) \wedge QC(\mathcal{F}) + \mu(1 - \mathcal{C}(\mathcal{F}))\,.$$

Based on Table 1, the possible values of the random vector $(\mathcal{C}(\mathcal{F}), \mathcal{C}(\mathcal{F}) \wedge QC(\mathcal{F}))$ are

$$(1,1)\,,\ (0,0)\,,\ (x_1,x_1)\,,\ (x_2,x_2)\,,\ (x_{12},\eta)\,,$$

where the value (x_{12}, η) is associated to the constituent $\bar{H}_1\bar{H}_2$. As we can see, $\mathcal{C}(\mathcal{F})$ and $\mathcal{C}(\mathcal{F}) \wedge QC(\mathcal{F})$ coincide when $H_1 \vee H_2$ is true; then, by Theorem 3, $x_{12} = \eta$, so that $\mathcal{C}(\mathcal{F})$ and $\mathcal{C}(\mathcal{F}) \wedge QC(\mathcal{F})$ coincide in all cases, that is $\mathcal{C}(\mathcal{F}) = \mathcal{C}(\mathcal{F}) \wedge QC(\mathcal{F})$. Then,

$$QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) = \mathcal{C}(\mathcal{F}) + \mu(1 - \mathcal{C}(\mathcal{F})) = \begin{cases} 1, & \text{if } \mathcal{C}(\mathcal{F}) = 1, \\ \mu, & \text{if } \mathcal{C}(\mathcal{F}) = 0, \\ x_1 + \mu(1 - x_1), & \text{if } \mathcal{C}(\mathcal{F}) = x_1, \\ x_2 + \mu(1 - x_2), & \text{if } \mathcal{C}(\mathcal{F}) = x_2, \\ x_{12} + \mu(1 - x_{12}), & \text{if } \mathcal{C}(\mathcal{F}) = x_{12}. \end{cases}$$

By the linearity of prevision, we obtain $\mu = x_{12} + \mu(1 - x_{12})$, that is $x_{12} = \mu x_{12}$. Then,

$$QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) = \mathcal{C}(\mathcal{F}) + \mu(1 - \mathcal{C}(\mathcal{F})) = \begin{cases} 1, & \text{if } \mathcal{C}(\mathcal{F}) = 1, \\ x_1 + \mu(1 - x_1), & \text{if } \mathcal{C}(\mathcal{F}) = x_1, \\ x_2 + \mu(1 - x_2), & \text{if } \mathcal{C}(\mathcal{F}) = x_2, \\ \mu, & \text{if } \mathcal{C}(\mathcal{F}) = 0, \text{ or } \mathcal{C}(\mathcal{F}) = x_{12}. \end{cases}$$

As $x_{12} = \mu x_{12}$ we immediately obtain that $\mu = 1$ when $x_{12} > 0$. We show below that $\mu = 1$ in all cases. We distinguish the following cases:
(a) $x_1 = x_2 = 0$; (b) $x_1 > 0, x_2 > 0$; (c) $x_1 = 0, x_2 > 0$; (d) $x_2 = 0, x_1 > 0$.
Case (a). Since $x_1 = x_2 = 0$, it holds that $x_1 + \mu(1 - x_1) = x_2 + \mu(1 - x_2) = \mu$, so that $QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) \in \{1, \mu\}$. Based on the betting scheme, $\mu = \mathbb{P}[QC(\mathcal{F})|\mathcal{C}(\mathcal{F})]$ is the amount to be paid in order to receive 1, or μ, according to whether the event $(\mathcal{C}(\mathcal{F}) = 1)$ is true, or false, respectively. Then, by coherence, it must be the case that $\mu = 1$. Therefore, $QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) = 1$.

Case (b). By coherence, μ must be a linear convex combination of the values 1, $x_1+\mu(1-x_1)$, and $x_2+\mu(1-x_2)$, that is,

$$\mu = \lambda_1 + \lambda_2(x_1 + \mu(1-x_1)) + \lambda_3(x_2 + \mu(1-x_2))\,, \qquad (14)$$

with $\lambda_h \geqslant 0, h = 1,2,3$, and $\lambda_1 + \lambda_2 + \lambda_3 = 1$. The equation (14) can be written as

$$\mu(\lambda_1 + \lambda_2 x_1 + \lambda_3 x_2) = \lambda_1 + \lambda_2 x_1 + \lambda_3 x_2\,,$$

where $\lambda_1+\lambda_2x_1+\lambda_3x_2 > 0$; then, $\mu = x_1+\mu(1-x_1) = x_2+\mu(1-x_2) = 1$ and $QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) = 1$.
Case (c). As $x_1 = 0$, it holds that $x_1 + \mu(1-x_1) = \mu$, so that

$$QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) \in \{1, x_2 + \mu(1-x_2), \mu\}\,.$$

Then, by coherence, μ must be a linear convex combination of the values $1, x_2 + \mu(1-x_2)$, that is

$$\mu = \lambda_1 + \lambda_2[x_2 + \mu(1-x_2)]\,,\ \lambda_1 + \lambda_2 = 1\,,\ \lambda_1 \geqslant 0\,,\ \lambda_2 \geqslant 0\,.$$

It follows that $\mu(\lambda_1+\lambda_2x_2) = \lambda_1+\lambda_2x_2$, with $\lambda_1+\lambda_2x_2 > 0$. Then, $\mu = 1$ and $QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) = 1$.
Case (d). As $x_2 = 0$, it holds that $x_2 + \mu(1-x_2) = \mu$, so that $QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) \in \{1, x_1 + \mu(1-x_1), \mu\}$. Then, by coherence, μ must be a linear convex combination of the values $1, x_1 + \mu(1-x_1)$, that is

$$\mu = \lambda_1 + \lambda_2[x_1 + \mu(1-x_1)]\,,\ \lambda_1 + \lambda_2 = 1\,,\ \lambda_1 \geqslant 0\,,\ \lambda_2 \geqslant 0\,.$$

It follows that $\mu(\lambda_1+\lambda_2x_1) = \lambda_1+\lambda_2x_1$, with $\lambda_1+\lambda_2x_1 > 0$. Then, $\mu = 1$ and $QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) = 1$.
Thus, from the p-consistency of the family $\mathcal{F}$ it follows that $QC(\mathcal{F})|\mathcal{C}(\mathcal{F}) = 1$.

□

Proof. of Theorem 8.
(⇒). We observe that by p-consistency $E_1H_1E_2H_2 \neq \emptyset$ and then $(E_1|H_1) \wedge (E_2|H_2) \neq 0$. By Theorem 1, $\{E_1|H_1, E_2|H_2\}$ p-entails $E_3|H_3$ if and only if it holds that $QC(\mathcal{S}) \subseteq E_3|H_3$ for some $\emptyset \neq \mathcal{S} \subseteq \{E_1|H_1, E_2|H_2\}$, or $H_3 \subseteq E_3$. We observe that, when $H_3 \nsubseteq E_3$, it holds that $\mathcal{S} = \{E_1|H_1\}$, or $\mathcal{S} = \{E_2|H_2\}$, or $\mathcal{S} = \{E_1|H_1, E_2|H_2\}$. We show that the iterated conditional may be represented as

$$(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = (E_1|H_1) \wedge (E_2|H_2) + \mu(1 - (E_1|H_1) \wedge (E_2|H_2))\,, \qquad (15)$$

where $\mu = \mathbb{P}[(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))]$.
We distinguish the following four cases:
(i) $H_3 \subseteq E_3$;
(ii) $H_3 \nsubseteq E_3$ and $E_1|H_1 \subseteq E_3|H_3$;
(iii) $H_3 \nsubseteq E_3$ and $E_2|H_2 \subseteq E_3|H_3$;
(iv) $H_3 \nsubseteq E_3$ and $QC(E_1|H_1, E_2|H_2) \subseteq E_3|H_3$.
Case (i). If $H_3 \subseteq E_3$, then $E_3|H_3 = P(E_3|H_3) = 1$. We set $P(E_i|H_i) = x_i$, $\mathbb{P}[(E_i|H_i) \wedge (E_j|H_j)] = x_{ij}$ and we recall that

$$\max\{x_i + x_j - 1, 0\} \leqslant x_{ij} \leqslant \min\{x_i, x_j\}\,.$$

Then, as $x_3 = 1$, we obtain $x_{13} = x_1$, $x_{23} = x_2$; it follows that for the random vector $((E_1|H_1) \wedge (E_2|H_2), (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3))$ the possible values are

$$(1,1)\,,\ (0,0)\,,\ (x_1,x_1)\,,\ (x_2,x_2)\,,\ (x_{12},x_{12})\,,\ (x_{12},x_{123})\,,$$

where $x_{123} = \mathbb{P}[(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)] = \mu$. As we can see, conditionally on $H_1 \vee H_2 \vee H_3$ being true, $(E_1|H_1) \wedge (E_2|H_2)$ and $(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)$ coincide; then, by Theorem 3, $x_{12} = x_{123}$, so that $(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)$ and $(E_1|H_1) \wedge (E_2|H_2)$ coincide. Then, (15) is satisfied.
Case (ii). As $E_1|H_1 \subseteq E_3|H_3$, by Proposition 1 it holds that $E_1|H_1 \wedge E_3|H_3 = E_1|H_1$ and $(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3) = (E_1|H_1) \wedge (E_2|H_2)$. Then, (15) is satisfied.
Case (iii). As $E_2|H_2 \subseteq E_3|H_3$, by Proposition 1 it holds that $E_2|H_2 \wedge E_3|H_3 = E_2|H_2$ and $(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3) = (E_1|H_1) \wedge (E_2|H_2)$. Then, (15) is satisfied.
Case (iv). By taking into account that $QC(E_1|H_1, E_2|H_2) \subseteq E_3|H_3$, the set of possible values of the random vector

$$((E_1|H_1) \wedge (E_2|H_2)\,,\, QC(E_1|H_1, E_2|H_2)\,,\, (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)),$$

as shown in Table 2, is

$$\{(1,1,1)\,,\,(0,0,0)\,,\,(x_1,1,x_1)\,,\,(x_2,1,x_2)\,,\,(x_{12},\nu_{12},x_{12})\,,\,(x_{12},\nu_{12},x_{123})\},$$

where $x_1 = P(E_1|H_1)$, $x_2 = P(E_2|H_2)$, $x_{12} = P[(E_1|H_1) \wedge (E_2|H_2)]$, $\nu_{12} = P[QC(E_1|H_1, E_2|H_2)]$, $x_{123} = \mathbb{P}[(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)]$. As we can see, conditionally on $H_1 \vee H_2 \vee H_3$ being true (i.e., $\bar{H}_1\bar{H}_2\bar{H}_3$ being false), $(E_1|H_1) \wedge (E_2|H_2)$ and $(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)$ coincide; then, by Theorem 3 it holds that $x_{12} = x_{123}$, so that $(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3) = (E_1|H_1) \wedge (E_2|H_2)$. Then, (15) is satisfied.

Now, by using the representation (15), for the iterated conditional we obtain

$$(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = \begin{cases} 1, & \text{if } E_1H_1E_2H_2 \text{ is true,} \\ \mu, & \text{if } \bar{E}_1H_1 \vee \bar{E}_2H_2 \text{ is true,} \\ x_1 + \mu(1-x_1), & \text{if } \bar{H}_1E_2H_2 \text{ is true,} \\ x_2 + \mu(1-x_2), & \text{if } E_1H_1\bar{H}_2 \text{ is true,} \\ x_{12} + \mu(1-x_{12}), & \text{if } \bar{H}_1\bar{H}_2 \text{ is true.} \end{cases} \tag{16}$$

Moreover, by the linearity of prevision it holds that

$$\mu = \mathbb{P}[(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))] = x_{12} + \mu(1 - x_{12})\,;$$

from which it follows that $x_{12} = \mu x_{12}$. Then, (16) becomes

$$(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = \begin{cases} 1, & \text{if } E_1H_1E_2H_2 \text{ is true,} \\ x_1 + \mu(1-x_1), & \text{if } \bar{H}_1E_2H_2 \text{ is true,} \\ x_2 + \mu(1-x_2), & \text{if } E_1H_1\bar{H}_2 \text{ is true,} \\ \mu, & \text{if } \bar{H}_1\bar{H}_2 \vee \bar{E}_1H_1 \vee \bar{E}_2H_2 \text{ is true.} \end{cases} \tag{17}$$

In order to prove that $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1$, as already done in the proof of Theorem 6, we distinguish the following cases: (a) $x_{12} > 0$; (b) $x_{12} = x_1 = x_2 = 0$; (c) $x_{12} = 0, x_1 > 0, x_2 > 0$; (d) $x_{12} = x_2 = 0, x_1 > 0$; (e) $x_{12} = x_1 = 0, x_2 > 0$.

C_h	$(E_1\|H_1) \wedge (E_2\|H_2)$	$QC(E_1\|H_1, E_2\|H_2)$	$(E_1\|H_1) \wedge (E_2\|H_2) \wedge (E_3\|H_3)$
$E_1H_1E_2H_2E_3H_3$	1	1	1
$E_1H_1\bar{E}_2H_2E_3H_3$	0	0	0
$E_1H_1\bar{E}_2H_2\bar{E}_3H_3$	0	0	0
$E_1H_1\bar{E}_2H_2\bar{H}_3$	0	0	0
$E_1H_1\bar{H}_2E_3H_3$	x_2	1	x_2
$\bar{E}_1H_1E_2H_2E_3H_3$	0	0	0
$\bar{E}_1H_1E_2H_2\bar{E}_3H_3$	0	0	0
$\bar{E}_1H_1E_2H_2\bar{H}_3$	0	0	0
$\bar{E}_1H_1\bar{E}_2H_2E_3H_3$	0	0	0
$\bar{E}_1H_1\bar{E}_2H_2\bar{E}_3H_3$	0	0	0
$\bar{E}_1H_1\bar{E}_2H_2\bar{H}_3$	0	0	0
$\bar{E}_1H_1\bar{H}_2E_3H_3$	0	0	0
$\bar{E}_1H_1\bar{H}_2\bar{E}_3H_3$	0	0	0
$\bar{E}_1H_1\bar{H}_2\bar{H}_3$	0	0	0
$\bar{H}_1E_2H_2E_3H_3$	x_1	1	x_1
$\bar{H}_1\bar{E}_2H_2E_3H_3$	0	0	0
$\bar{H}_1\bar{E}_2H_2\bar{E}_3H_3$	0	0	0
$\bar{H}_1\bar{E}_2H_2\bar{H}_3$	0	0	0
$\bar{H}_1\bar{H}_2E_3H_3$	x_{12}	ν_{12}	x_{12}
$\bar{H}_1\bar{H}_2\bar{H}_3$	x_{12}	ν_{12}	x_{123}

Table 2: Possible values of the random vector $((E_1|H_1) \wedge (E_2|H_2),\ QC(E_1|H_1, E_2|H_2),$ $(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3))$, under the assumption that $QC(E_1|H_1, E_2|H_2) \subseteq E_3|H_3$.

Case (a). As $x_{12} > 0$ and $x_{12} = \mu x_{12}$, it follows that $\mu = 1$ and then $x_1 + \mu(1 - x_1) = x_2 + \mu(1 - x_2) = 1$. Therefore, $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1$.
Case (b). As $x_1 = x_2 = 0$, it holds that $x_1 + \mu(1 - x_1) = x_2 + \mu(1 - x_2) = \mu$, so that $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) \in \{1, \mu\}$. We observe that, based on the metaphor of the betting scheme, $\mu = \mathbb{P}[(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))]$ is the amount to be paid in order to receive 1, or μ, according to whether $E_1H_1E_2H_2$ is true, or false, respectively. Then, by discarding the case where it is received back what has been paid, coherence requires that $\mu = 1$. Therefore $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1$.
Case (c). By coherence, μ must be a linear convex combination of the values 1, $x_1+\mu(1-x_1)$, and $x_2 + \mu(1 - x_2)$, that is,

$$\mu = \lambda_1 + \lambda_2(x_1 + \mu(1 - x_1)) + \lambda_3(x_2 + \mu(1 - x_2))\,, \tag{18}$$

with $\lambda_h \geqslant 0, h = 1, 2, 3$, and $\lambda_1 + \lambda_2 + \lambda_3 = 1$. The equation (18) can be written as

$$\mu(\lambda_1 + \lambda_2 x_1 + \lambda_3 x_2) = \lambda_1 + \lambda_2 x_1 + \lambda_3 x_2\,,$$

where $\lambda_1 + \lambda_2 x_1 + \lambda_3 x_2 > 0$; then, $\mu = x_1 + \mu(1 - x_1) = x_2 + \mu(1 - x_2) = 1$ and $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1$.
Case (d). As $x_2 = 0$, it holds that $x_2 + \mu(1 - x_2) = \mu$, so that

$$(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) \in \{1, x_1 + \mu(1 - x_1), \mu\}\,.$$

Then, by coherence, μ must be a linear convex combination of the values $1, x_1 + \mu(1 - x_1)$, that is

$$\mu = \lambda_1 + \lambda_2[x_1 + \mu(1 - x_1)]\,,\ \lambda_1 + \lambda_2 = 1\,,\ \lambda_1 \geqslant 0\,,\ \lambda_2 \geqslant 0\,.$$

It follows that $\mu(\lambda_1 + \lambda_2 x_1) = \lambda_1 + \lambda_2 x_1$, with $\lambda_1 + \lambda_2 x_1 > 0$. Then, $\mu = 1$ and $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1$.
Case (e). As $x_1 = 0$, it holds that $x_1 + \mu(1 - x_1) = \mu$, so that

$$(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) \in \{1, x_2 + \mu(1 - x_2), \mu\}\,.$$

Then, by coherence, μ must be a linear convex combination of the values $1, x_2 + \mu(1 - x_2)$, that is

$$\mu = \lambda_1 + \lambda_2[x_2 + \mu(1 - x_2)]\,,\ \lambda_1 + \lambda_2 = 1\,,\ \lambda_1 \geqslant 0\,,\ \lambda_2 \geqslant 0\,.$$

It follows that $\mu(\lambda_1 + \lambda_2 x_2) = \lambda_1 + \lambda_2 x_2$, with $\lambda_1 + \lambda_2 x_2 > 0$. Then, $\mu = 1$ and $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1$.
($\Leftarrow$). Assume that $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1$, so that the unique coherent prevision assessment on $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))$ is $\mu = 1$. From Remark 2 it holds that $x_{123} = \mu x_{12} = x_{12}$. Moreover, $x_{123} \leqslant x_3$ (Gilio & Sanfilippo, 2017, Equation (8)) and $x_{12} \geqslant \max\{x_1 + x_2 - 1, 0\}$ (see Equation (4)). Then, it holds that

$$\max\{x_1 + x_2 - 1,\, 0\} \ \leqslant\ x_{12} \ =\ x_{123} \ \leqslant\ x_3\,,$$

and, when $x_1 = x_2 = 1$, it follows that $x_{12} = x_{123} = x_3 = 1$. Therefore, $\{E_1|H_1, E_2|H_2\}$ p-entails $E_3|H_3$.

□

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